Preparation

Remark on "replace" (->) function of Mathematica

Defining “a^2 -> 1” has no effect on a^3, so that replace should be defined for every higher - order terms.
Example:

```
1 + 2 a + 3 a^2 + 4 a^3 /. a^2 → 1
```

$4 + 2\,a + 4\,a^3$

Defining pattern function "(a+b)(a-b)=a^2-b^2" is useful

```
MyProd[p1_, p2_] := Module[{a, b},
    a = Factor[(p1 + p2) / 2];
    b = Factor[(p1 - p2) / 2];
    a^2 - b^2
  ];
```

Example :

```
MyProd[a + b + c + d, a - b + c - d]
```

$(a + c)^2 - (b + d)^2$

Mathematica is weak in this pattern. Example :

```
Expand[(a + b + c + d) (a - b + c - d)]
```

$a^2 - b^2 + 2\,a\,c + c^2 - 2\,b\,d - d^2$

```
Simplify[(a + b + c + d) (a - b + c - d)]
```

$(a - b + c - d)\,(a + b + c + d)$

The result of MyProd[] is not obtained by default. MyProd[] also works for numbers :

```
MyProd[3, 5]
```

15

Norm expression when e = 2

Let ζ be (ζ^4 = -1, ζ^8 = 1). We define "Replace" for it.

```
ruleζ1 = Flatten[Table[ζ^(i+8 j) -> ζ^i, {j, 1, 3}, {i, 0, 7}]]
```

$\{\zeta^8 \to 1, \zeta^9 \to \zeta, \zeta^{10} \to \zeta^2, \zeta^{11} \to \zeta^3, \zeta^{12} \to \zeta^4, \zeta^{13} \to \zeta^5, \zeta^{14} \to \zeta^6, \zeta^{15} \to \zeta^7,$
$\zeta^{16} \to 1, \zeta^{17} \to \zeta, \zeta^{18} \to \zeta^2, \zeta^{19} \to \zeta^3, \zeta^{20} \to \zeta^4, \zeta^{21} \to \zeta^5, \zeta^{22} \to \zeta^6, \zeta^{23} \to \zeta^7,$
$\zeta^{24} \to 1, \zeta^{25} \to \zeta, \zeta^{26} \to \zeta^2, \zeta^{27} \to \zeta^3, \zeta^{28} \to \zeta^4, \zeta^{29} \to \zeta^5, \zeta^{30} \to \zeta^6, \zeta^{31} \to \zeta^7\}$

```
ruleζ2 = Table[ζ^i → -ζ^(i-4), {i, 4, 7}]
```

$\{\zeta^4 \to -1, \zeta^5 \to -\zeta, \zeta^6 \to -\zeta^2, \zeta^7 \to -\zeta^3\}$

Polynomials of ζ:

```
p[r_] := (Sum[a_i ζ^(i r), {i, 0, 2^2-1}]) /. ruleζ1 /. ruleζ2;
```

```
p1 = p[1]
```

$a_0 + \zeta\, a_1 + \zeta^2\, a_2 + \zeta^3\, a_3$

```
p3 = p[3]
```

$a_0 + \zeta^3\, a_1 - \zeta^2\, a_2 + \zeta\, a_3$

```
p5 = p[5]
```

$a_0 - \zeta\, a_1 + \zeta^2\, a_2 - \zeta^3\, a_3$

```
p7 = p[7]
```

$a_0 - \zeta^3\, a_1 - \zeta^2\, a_2 - \zeta\, a_3$

Simple product is not good as follows.

```
Expand[p1 p3 p5 p7] /. ruleζ1 /. ruleζ2
```

$a_0^4 + a_1^4 - 4\, a_0\, a_1^2\, a_2 + 2\, a_0^2\, a_2^2 + a_2^4 + 4\, a_0^2\, a_1\, a_3 - 4\, a_1\, a_2^2\, a_3 + 2\, a_1^2\, a_3^2 + 4\, a_0\, a_2\, a_3^2 + a_3^4$

```
Simplify[%]
```

$a_0^4 + a_1^4 + a_2^4 - 4\, a_1\, a_2^2\, a_3 + 2\, a_1^2\, a_3^2 + a_3^4 + 2\, a_0^2\, \left(a_2^2 + 2\, a_1\, a_3\right) + 4\, a_0\, a_2\, \left(-a_1^2 + a_3^2\right)$

We use MyProd[] as follows.

```
MyProd[p1, p5]
```

$\left(a_0 + \zeta^2\, a_2\right)^2 - \zeta^2\, \left(a_1 + \zeta^2\, a_3\right)^2$

```
MyProd[p3, p7]
```

$\left(a_0 - \zeta^2\, a_2\right)^2 - \zeta^2\, \left(\zeta^2\, a_1 + a_3\right)^2$

Let η be (η^2 = -1, η^4 = 1). We replace ζ^2 to η .

```
ruleζη = ζ^2 -> η;
```

```
prod15 = MyProd[p1, p5] /. ruleζη
```

$(a_0 + \eta\, a_2)^2 - \eta\, (a_1 + \eta\, a_3)^2$

```
prod37 = MyProd[p3, p7] /. ruleζη
```

$(a_0 - \eta\, a_2)^2 - \eta\, (\eta\, a_1 + a_3)^2$

We define "Replace" for η.

```
ruleη1 = Flatten[Table[η^(i+4 j) → η^i, {j, 1, 1}, {i, 0, 3}]]
```

$\{\eta^4 \to 1,\ \eta^5 \to \eta,\ \eta^6 \to \eta^2,\ \eta^7 \to \eta^3\}$

```
ruleη2 = Table[η^i → -η^(i-2), {i, 2, 3}]
```

$\{\eta^2 \to -1,\ \eta^3 \to -\eta\}$

```
prod15mfd = Collect[Expand[prod15] /. ruleη1 /. ruleη2, η]
```

$a_0^2 - a_2^2 + 2\, a_1\, a_3 + \eta\, \left(-a_1^2 + 2\, a_0\, a_2 + a_3^2\right)$

```
prod37mfd = Collect[Expand[prod37] /. ruleη1 /. ruleη2, η]
```

$a_0^2 - a_2^2 + 2\, a_1\, a_3 + \eta\, \left(a_1^2 - 2\, a_0\, a_2 - a_3^2\right)$

```
norm2 = MyProd[prod15mfd, prod37mfd] /. ruleη2
```

$\left(a_0^2 - a_2^2 + 2\, a_1\, a_3\right)^2 + \left(a_1^2 - 2\, a_0\, a_2 - a_3^2\right)^2$

Clearly, we see norm2 is a nonnegative real number.

Norm expression when e = 3

Let ω be (ω^8 = -1, ω^16 = 1). We define "Replace" for it.

```
ruleω1 = Flatten[Table[ω^(i+16 j) -> ω^i, {j, 1, 7}, {i, 0, 15}]]
```

$\{\omega^{16} \to 1,\ \omega^{17} \to \omega,\ \omega^{18} \to \omega^2,\ \omega^{19} \to \omega^3,\ \omega^{20} \to \omega^4,\ \omega^{21} \to \omega^5,\ \omega^{22} \to \omega^6,\ \omega^{23} \to \omega^7,\ \omega^{24} \to \omega^8,\ \omega^{25} \to \omega^9,\ \omega^{26} \to \omega^{10},\ \omega^{27} \to \omega^{11},\ \omega^{28} \to \omega^{12},\ \omega^{29} \to \omega^{13},\ \omega^{30} \to \omega^{14},\ \omega^{31} \to \omega^{15},\ \omega^{32} \to 1,\ \omega^{33} \to \omega,\ \omega^{34} \to \omega^2,\ \omega^{35} \to \omega^3,\ \omega^{36} \to \omega^4,\ \omega^{37} \to \omega^5,\ \omega^{38} \to \omega^6,\ \omega^{39} \to \omega^7,\ \omega^{40} \to \omega^8,\ \omega^{41} \to \omega^9,\ \omega^{42} \to \omega^{10},\ \omega^{43} \to \omega^{11},\ \omega^{44} \to \omega^{12},\ \omega^{45} \to \omega^{13},\ \omega^{46} \to \omega^{14},\ \omega^{47} \to \omega^{15},\ \omega^{48} \to 1,\ \omega^{49} \to \omega,\ \omega^{50} \to \omega^2,\ \omega^{51} \to \omega^3,\ \omega^{52} \to \omega^4,\ \omega^{53} \to \omega^5,\ \omega^{54} \to \omega^6,\ \omega^{55} \to \omega^7,\ \omega^{56} \to \omega^8,\ \omega^{57} \to \omega^9,\ \omega^{58} \to \omega^{10},\ \omega^{59} \to \omega^{11},\ \omega^{60} \to \omega^{12},\ \omega^{61} \to \omega^{13},\ \omega^{62} \to \omega^{14},\ \omega^{63} \to \omega^{15},\ \omega^{64} \to 1,\ \omega^{65} \to \omega,\ \omega^{66} \to \omega^2,\ \omega^{67} \to \omega^3,\ \omega^{68} \to \omega^4,\ \omega^{69} \to \omega^5,\ \omega^{70} \to \omega^6,\ \omega^{71} \to \omega^7,\ \omega^{72} \to \omega^8,\ \omega^{73} \to \omega^9,\ \omega^{74} \to \omega^{10},\ \omega^{75} \to \omega^{11},\ \omega^{76} \to \omega^{12},\ \omega^{77} \to \omega^{13},\ \omega^{78} \to \omega^{14},\ \omega^{79} \to \omega^{15},\ \omega^{80} \to 1,\ \omega^{81} \to \omega,\ \omega^{82} \to \omega^2,\ \omega^{83} \to \omega^3,\ \omega^{84} \to \omega^4,\ \omega^{85} \to \omega^5,\ \omega^{86} \to \omega^6,\ \omega^{87} \to \omega^7,\ \omega^{88} \to \omega^8,\ \omega^{89} \to \omega^9,\ \omega^{90} \to \omega^{10},\ \omega^{91} \to \omega^{11},\ \omega^{92} \to \omega^{12},\ \omega^{93} \to \omega^{13},\ \omega^{94} \to \omega^{14},\ \omega^{95} \to \omega^{15},\ \omega^{96} \to 1,\ \omega^{97} \to \omega,\ \omega^{98} \to \omega^2,\ \omega^{99} \to \omega^3,\ \omega^{100} \to \omega^4,\ \omega^{101} \to \omega^5,\ \omega^{102} \to \omega^6,\ \omega^{103} \to \omega^7,\ \omega^{104} \to \omega^8,\ \omega^{105} \to \omega^9,\ \omega^{106} \to \omega^{10},\ \omega^{107} \to \omega^{11},\ \omega^{108} \to \omega^{12},\ \omega^{109} \to \omega^{13},\ \omega^{110} \to \omega^{14},\ \omega^{111} \to \omega^{15},\ \omega^{112} \to 1,\ \omega^{113} \to \omega,\ \omega^{114} \to \omega^2,\ \omega^{115} \to \omega^3,\ \omega^{116} \to \omega^4,\ \omega^{117} \to \omega^5,\ \omega^{118} \to \omega^6,\ \omega^{119} \to \omega^7,\ \omega^{120} \to \omega^8,\ \omega^{121} \to \omega^9,\ \omega^{122} \to \omega^{10},\ \omega^{123} \to \omega^{11},\ \omega^{124} \to \omega^{12},\ \omega^{125} \to \omega^{13},\ \omega^{126} \to \omega^{14},\ \omega^{127} \to \omega^{15}\}$

```
ruleω2 = Table[ω^i → -ω^(i-8), {i, 8, 15}]
```

$\{\omega^8 \to -1,\ \omega^9 \to -\omega,\ \omega^{10} \to -\omega^2,\ \omega^{11} \to -\omega^3,\ \omega^{12} \to -\omega^4,\ \omega^{13} \to -\omega^5,\ \omega^{14} \to -\omega^6,\ \omega^{15} \to -\omega^7\}$

$$\texttt{q[r\_] :=} \left(\sum_{i=0}^{2^3-1} a_i\, \omega^{i\, r}\right) \texttt{ /. rule}\omega\texttt{1 /. rule}\omega\texttt{2;}$$

```
q1 = q[1]
```

$a_0 + \omega\, a_1 + \omega^2\, a_2 + \omega^3\, a_3 + \omega^4\, a_4 + \omega^5\, a_5 + \omega^6\, a_6 + \omega^7\, a_7$

```
q3 = q[3]
```

$a_0 + \omega^3\, a_1 + \omega^6\, a_2 - \omega\, a_3 - \omega^4\, a_4 - \omega^7\, a_5 + \omega^2\, a_6 + \omega^5\, a_7$

```
q5 = q[5]
```

$a_0 + \omega^5\, a_1 - \omega^2\, a_2 - \omega^7\, a_3 + \omega^4\, a_4 - \omega\, a_5 - \omega^6\, a_6 + \omega^3\, a_7$

```
q7 = q[7]
```

$a_0 + \omega^7\, a_1 - \omega^6\, a_2 + \omega^5\, a_3 - \omega^4\, a_4 + \omega^3\, a_5 - \omega^2\, a_6 + \omega\, a_7$

```
q9 = q[9]
```

$a_0 - \omega\, a_1 + \omega^2\, a_2 - \omega^3\, a_3 + \omega^4\, a_4 - \omega^5\, a_5 + \omega^6\, a_6 - \omega^7\, a_7$

```
q11 = q[11]
```

$$a_0 - \omega^3 a_1 + \omega^6 a_2 + \omega a_3 - \omega^4 a_4 + \omega^7 a_5 + \omega^2 a_6 - \omega^5 a_7$$

```
q13 = q[13]
```

$$a_0 - \omega^5 a_1 - \omega^2 a_2 + \omega^7 a_3 + \omega^4 a_4 + \omega a_5 - \omega^6 a_6 - \omega^3 a_7$$

```
q15 = q[15]
```

$$a_0 - \omega^7 a_1 - \omega^6 a_2 - \omega^5 a_3 - \omega^4 a_4 - \omega^3 a_5 - \omega^2 a_6 - \omega a_7$$

```
MyProd[q1, q9]
```

$$\left(a_0 + \omega^2 a_2 + \omega^4 a_4 + \omega^6 a_6\right)^2 - \omega^2 \left(a_1 + \omega^2 a_3 + \omega^4 a_5 + \omega^6 a_7\right)^2$$

We see that translation ω^2 to ζ after ' MyProd ' is effective.

```
ruleωζ = Table[ω^(2 j) → ζ^j, {j, 1, 3}]
```

$$\left\{\omega^2 \to \zeta,\ \omega^4 \to \zeta^2,\ \omega^6 \to \zeta^3\right\}$$

```
prod1et9 = MyProd[q1, q9] /. ruleωζ
```

$$\left(a_0 + \zeta a_2 + \zeta^2 a_4 + \zeta^3 a_6\right)^2 - \zeta \left(a_1 + \zeta a_3 + \zeta^2 a_5 + \zeta^3 a_7\right)^2$$

```
prod3et11 = MyProd[q3, q11] /. ruleωζ
```

$$\left(a_0 + \zeta^3 a_2 - \zeta^2 a_4 + \zeta a_6\right)^2 - \zeta \left(-\zeta a_1 + a_3 + \zeta^3 a_5 - \zeta^2 a_7\right)^2$$

```
prod5et13 = MyProd[q5, q13] /. ruleωζ
```

$$\left(a_0 - \zeta a_2 + \zeta^2 a_4 - \zeta^3 a_6\right)^2 - \zeta \left(-\zeta^2 a_1 + \zeta^3 a_3 + a_5 - \zeta a_7\right)^2$$

```
prod7et15 = MyProd[q7, q15] /. ruleωζ
```

$$\left(a_0 - \zeta^3 a_2 - \zeta^2 a_4 - \zeta a_6\right)^2 - \zeta \left(\zeta^3 a_1 + \zeta^2 a_3 + \zeta a_5 + a_7\right)^2$$

We simplify these by applying the rule of ζ as same in the case of e = 2.

```
prod1et9mfd = Collect[Expand[prod1et9] /. ruleζ1 /. ruleζ2, ζ]
```

$$a_0^2 - a_4^2 + 2 a_3 a_5 - 2 a_2 a_6 + 2 a_1 a_7 + \zeta \left(-a_1^2 + 2 a_0 a_2 + a_5^2 - 2 a_4 a_6 + 2 a_3 a_7\right) + \zeta^2 \left(a_2^2 - 2 a_1 a_3 + 2 a_0 a_4 - a_6^2 + 2 a_5 a_7\right) + \zeta^3 \left(-a_3^2 + 2 a_2 a_4 - 2 a_1 a_5 + 2 a_0 a_6 + a_7^2\right)$$

```
prod3et11mfd = Collect[Expand[prod3et11] /. ruleζ1 /. ruleζ2, ζ]
```

$$a_0^2 - a_4^2 + 2 a_3 a_5 - 2 a_2 a_6 + 2 a_1 a_7 + \zeta^3 \left(-a_1^2 + 2 a_0 a_2 + a_5^2 - 2 a_4 a_6 + 2 a_3 a_7\right) + \zeta^2 \left(-a_2^2 + 2 a_1 a_3 - 2 a_0 a_4 + a_6^2 - 2 a_5 a_7\right) + \zeta \left(-a_3^2 + 2 a_2 a_4 - 2 a_1 a_5 + 2 a_0 a_6 + a_7^2\right)$$

```
prod5et13mfd = Collect[Expand[prod5et13] /. ruleζ1 /. ruleζ2, ζ]
```

$$a_0^2 - a_4^2 + 2 a_3 a_5 - 2 a_2 a_6 + 2 a_1 a_7 + \zeta \left(a_1^2 - 2 a_0 a_2 - a_5^2 + 2 a_4 a_6 - 2 a_3 a_7\right) + \zeta^2 \left(a_2^2 - 2 a_1 a_3 + 2 a_0 a_4 - a_6^2 + 2 a_5 a_7\right) + \zeta^3 \left(a_3^2 - 2 a_2 a_4 + 2 a_1 a_5 - 2 a_0 a_6 - a_7^2\right)$$

```
prod7et15mfd = Collect[Expand[prod7et15] /. ruleζ1 /. ruleζ2, ζ]
```

$$a_0^2 - a_4^2 + 2 a_3 a_5 - 2 a_2 a_6 + 2 a_1 a_7 + \zeta^3 \left(a_1^2 - 2 a_0 a_2 - a_5^2 + 2 a_4 a_6 - 2 a_3 a_7\right) + \zeta^2 \left(-a_2^2 + 2 a_1 a_3 - 2 a_0 a_4 + a_6^2 - 2 a_5 a_7\right) + \zeta \left(a_3^2 - 2 a_2 a_4 + 2 a_1 a_5 - 2 a_0 a_6 - a_7^2\right)$$

```
prod1e5e9e13 = MyProd[prod1et9mfd, prod5et13mfd] /. ruleζη
```

$$\left(a_0^2 + \eta a_2^2 - 2 \eta a_1 a_3 + 2 \eta a_0 a_4 - a_4^2 + 2 a_3 a_5 - 2 a_2 a_6 - \eta a_6^2 + 2 a_1 a_7 + 2 \eta a_5 a_7\right)^2 - \eta \left(a_1^2 - 2 a_0 a_2 + \eta a_3^2 - 2 \eta a_2 a_4 + 2 \eta a_1 a_5 - a_5^2 - 2 \eta a_0 a_6 + 2 a_4 a_6 - 2 a_3 a_7 - \eta a_7^2\right)^2$$

```
prod3e7e11e15 = MyProd[prod3et11mfd, prod7et15mfd] /. ruleξη
```

$$\begin{aligned}&\left(a_0^2 - \eta\, a_2^2 + 2\,\eta\, a_1 a_3 - 2\,\eta\, a_0 a_4 - a_4^2 + 2\, a_3 a_5 - 2\, a_2 a_6 + \eta\, a_6^2 + 2\, a_1 a_7 - 2\,\eta\, a_5 a_7\right)^2 -\\ &\quad \eta\left(\eta\, a_1^2 - 2\,\eta\, a_0 a_2 + a_3^2 - 2\, a_2 a_4 + 2\, a_1 a_5 - \eta\, a_5^2 - 2\, a_0 a_6 + 2\,\eta\, a_4 a_6 - 2\,\eta\, a_3 a_7 - a_7^2\right)^2\end{aligned}$$

```
prod1e5e9e13mfd = Collect[Expand[prod1e5e9e13] /. ruleη1 /. ruleη2, η]
```

$$\begin{aligned}&a_0^4 - a_2^4 + 4\, a_1 a_2^2 a_3 - 2\, a_1^2 a_3^2 - 4\, a_0 a_2 a_3^2 - 4\, a_1^2 a_2 a_4 + 4\, a_0 a_2^2 a_4 + 8\, a_0 a_1 a_3 a_4 -\\
&\quad 6\, a_0^2 a_4^2 + a_4^4 + 4\, a_1^3 a_5 - 8\, a_0 a_1 a_2 a_5 + 4\, a_0^2 a_3 a_5 - 4\, a_3 a_4^2 a_5 + 2\, a_3^2 a_5^2 + 4\, a_2 a_4 a_5^2 -\\
&\quad 4\, a_1 a_5^3 - 4\, a_0 a_1^2 a_6 + 4\, a_0^2 a_2 a_6 + 4\, a_3^2 a_4 a_6 - 4\, a_2 a_4^2 a_6 - 8\, a_2 a_3 a_5 a_6 +\\
&\quad 8\, a_1 a_4 a_5 a_6 + 4\, a_0 a_5^2 a_6 + 6\, a_2^2 a_6^2 - 4\, a_1 a_3 a_6^2 - 4\, a_0 a_4 a_6^2 - a_6^4 + 4\, a_0^2 a_1 a_7 - 4\, a_3^3 a_7 +\\
&\quad 8\, a_2 a_3 a_4 a_7 - 4\, a_1 a_4^2 a_7 - 4\, a_2^2 a_5 a_7 + 8\, a_1 a_3 a_5 a_7 - 8\, a_0 a_4 a_5 a_7 - 8\, a_1 a_2 a_6 a_7 +\\
&\quad 8\, a_0 a_3 a_6 a_7 + 4\, a_5 a_6^2 a_7 + 2\, a_1^2 a_7^2 + 4\, a_0 a_2 a_7^2 - 2\, a_5^2 a_7^2 - 4\, a_4 a_6 a_7^2 + 4\, a_3 a_7^3 +\\
&\quad \eta\,\big(-a_1^4 + 4\, a_0 a_1^2 a_2 - 2\, a_0^2 a_2^2 - 4\, a_0^2 a_1 a_3 + a_3^4 + 4\, a_0^3 a_4 - 4\, a_2 a_3^2 a_4 + 2\, a_2^2 a_4^2 + 4\, a_1 a_3 a_4^2 - 4\, a_0 a_4^3 +\\
&\qquad 4\, a_2^2 a_3 a_5 - 4\, a_1 a_3^2 a_5 - 8\, a_1 a_2 a_4 a_5 + 8\, a_0 a_3 a_4 a_5 + 6\, a_1^2 a_5^2 - 4\, a_0 a_2 a_5^2 - a_5^4 -\\
&\qquad 4\, a_2^3 a_6 + 8\, a_1 a_2 a_3 a_6 - 4\, a_0 a_3^2 a_6 - 4\, a_1^2 a_4 a_6 + 8\, a_0 a_2 a_4 a_6 - 8\, a_0 a_1 a_5 a_6 +\\
&\qquad 4\, a_4 a_5^2 a_6 + 2\, a_0^2 a_6^2 - 2\, a_4^2 a_6^2 - 4\, a_3 a_5 a_6^2 + 4\, a_2 a_6^3 + 4\, a_1 a_2^2 a_7 - 4\, a_1^2 a_3 a_7 -\\
&\qquad 8\, a_0 a_2 a_3 a_7 + 8\, a_0 a_1 a_4 a_7 + 4\, a_0^2 a_5 a_7 - 4\, a_4^2 a_5 a_7 + 4\, a_3 a_5^2 a_7 + 8\, a_3 a_4 a_6 a_7 -\\
&\qquad 8\, a_2 a_5 a_6 a_7 - 4\, a_1 a_6^2 a_7 - 6\, a_3^2 a_7^2 + 4\, a_2 a_4 a_7^2 + 4\, a_1 a_5 a_7^2 + 4\, a_0 a_6 a_7^2 + a_7^4\big)\end{aligned}$$

```
prod3e7e11e15mfd = Collect[Expand[prod3e7e11e15] /. ruleη1 /. ruleη2, η]
```

$$\begin{aligned}&a_0^4 - a_2^4 + 4\, a_1 a_2^2 a_3 - 2\, a_1^2 a_3^2 - 4\, a_0 a_2 a_3^2 - 4\, a_1^2 a_2 a_4 + 4\, a_0 a_2^2 a_4 + 8\, a_0 a_1 a_3 a_4 -\\
&\quad 6\, a_0^2 a_4^2 + a_4^4 + 4\, a_1^3 a_5 - 8\, a_0 a_1 a_2 a_5 + 4\, a_0^2 a_3 a_5 - 4\, a_3 a_4^2 a_5 + 2\, a_3^2 a_5^2 + 4\, a_2 a_4 a_5^2 -\\
&\quad 4\, a_1 a_5^3 - 4\, a_0 a_1^2 a_6 + 4\, a_0^2 a_2 a_6 + 4\, a_3^2 a_4 a_6 - 4\, a_2 a_4^2 a_6 - 8\, a_2 a_3 a_5 a_6 +\\
&\quad 8\, a_1 a_4 a_5 a_6 + 4\, a_0 a_5^2 a_6 + 6\, a_2^2 a_6^2 - 4\, a_1 a_3 a_6^2 - 4\, a_0 a_4 a_6^2 - a_6^4 + 4\, a_0^2 a_1 a_7 - 4\, a_3^3 a_7 +\\
&\quad 8\, a_2 a_3 a_4 a_7 - 4\, a_1 a_4^2 a_7 - 4\, a_2^2 a_5 a_7 + 8\, a_1 a_3 a_5 a_7 - 8\, a_0 a_4 a_5 a_7 - 8\, a_1 a_2 a_6 a_7 +\\
&\quad 8\, a_0 a_3 a_6 a_7 + 4\, a_5 a_6^2 a_7 + 2\, a_1^2 a_7^2 + 4\, a_0 a_2 a_7^2 - 2\, a_5^2 a_7^2 - 4\, a_4 a_6 a_7^2 + 4\, a_3 a_7^3 +\\
&\quad \eta\,\big(a_1^4 - 4\, a_0 a_1^2 a_2 + 2\, a_0^2 a_2^2 + 4\, a_0^2 a_1 a_3 - a_3^4 - 4\, a_0^3 a_4 + 4\, a_2 a_3^2 a_4 - 2\, a_2^2 a_4^2 - 4\, a_1 a_3 a_4^2 + 4\, a_0 a_4^3 -\\
&\qquad 4\, a_2^2 a_3 a_5 + 4\, a_1 a_3^2 a_5 + 8\, a_1 a_2 a_4 a_5 - 8\, a_0 a_3 a_4 a_5 - 6\, a_1^2 a_5^2 + 4\, a_0 a_2 a_5^2 + a_5^4 +\\
&\qquad 4\, a_2^3 a_6 - 8\, a_1 a_2 a_3 a_6 + 4\, a_0 a_3^2 a_6 + 4\, a_1^2 a_4 a_6 - 8\, a_0 a_2 a_4 a_6 + 8\, a_0 a_1 a_5 a_6 -\\
&\qquad 4\, a_4 a_5^2 a_6 - 2\, a_0^2 a_6^2 + 2\, a_4^2 a_6^2 + 4\, a_3 a_5 a_6^2 - 4\, a_2 a_6^3 - 4\, a_1 a_2^2 a_7 + 4\, a_1^2 a_3 a_7 +\\
&\qquad 8\, a_0 a_2 a_3 a_7 - 8\, a_0 a_1 a_4 a_7 - 4\, a_0^2 a_5 a_7 + 4\, a_4^2 a_5 a_7 - 4\, a_3 a_5^2 a_7 - 8\, a_3 a_4 a_6 a_7 +\\
&\qquad 8\, a_2 a_5 a_6 a_7 + 4\, a_1 a_6^2 a_7 + 6\, a_3^2 a_7^2 - 4\, a_2 a_4 a_7^2 - 4\, a_1 a_5 a_7^2 - 4\, a_0 a_6 a_7^2 - a_7^4\big)\end{aligned}$$

Numbers of terms in the coefficient polynomials are :

```
Table[Length[Coefficient[prod1e5e9e13mfd, η, i]], {i, 0, 1}]
```

{43, 43}

The maximum value of the coefficients of those terms is 8 as following:

```
tmp = Coefficient[prod1e5e9e13mfd, η, 0]
```

$$\begin{aligned}&a_0^4 - a_2^4 + 4\, a_1 a_2^2 a_3 - 2\, a_1^2 a_3^2 - 4\, a_0 a_2 a_3^2 - 4\, a_1^2 a_2 a_4 + 4\, a_0 a_2^2 a_4 + 8\, a_0 a_1 a_3 a_4 -\\
&\quad 6\, a_0^2 a_4^2 + a_4^4 + 4\, a_1^3 a_5 - 8\, a_0 a_1 a_2 a_5 + 4\, a_0^2 a_3 a_5 - 4\, a_3 a_4^2 a_5 + 2\, a_3^2 a_5^2 + 4\, a_2 a_4 a_5^2 -\\
&\quad 4\, a_1 a_5^3 - 4\, a_0 a_1^2 a_6 + 4\, a_0^2 a_2 a_6 + 4\, a_3^2 a_4 a_6 - 4\, a_2 a_4^2 a_6 - 8\, a_2 a_3 a_5 a_6 +\\
&\quad 8\, a_1 a_4 a_5 a_6 + 4\, a_0 a_5^2 a_6 + 6\, a_2^2 a_6^2 - 4\, a_1 a_3 a_6^2 - 4\, a_0 a_4 a_6^2 - a_6^4 + 4\, a_0^2 a_1 a_7 - 4\, a_3^3 a_7 +\\
&\quad 8\, a_2 a_3 a_4 a_7 - 4\, a_1 a_4^2 a_7 - 4\, a_2^2 a_5 a_7 + 8\, a_1 a_3 a_5 a_7 - 8\, a_0 a_4 a_5 a_7 - 8\, a_1 a_2 a_6 a_7 +\\
&\quad 8\, a_0 a_3 a_6 a_7 + 4\, a_5 a_6^2 a_7 + 2\, a_1^2 a_7^2 + 4\, a_0 a_2 a_7^2 - 2\, a_5^2 a_7^2 - 4\, a_4 a_6 a_7^2 + 4\, a_3 a_7^3\end{aligned}$$

```
Abs[MonomialList[tmp] /. Table[a_i → 1, {i, 0, 7}]]
```

{1, 4, 4, 4, 6, 4, 8, 4, 4, 4, 8, 8, 8, 4, 4, 1, 4, 4, 6, 4,
 8, 8, 8, 4, 4, 4, 2, 2, 8, 4, 4, 8, 4, 4, 4, 2, 4, 4, 1, 4, 2, 4, 1}

```
Max[%]
```

8

```
norm3 = MyProd[prod1e5e9e13mfd, prod3e7e11e15mfd] /. ruleη2
```

$$\begin{aligned}&\left(a_0^4 - a_2^4 + 4\,a_1\,a_2^2\,a_3 - 2\,a_1^2\,a_3^2 - 4\,a_0\,a_2\,a_3^2 - 4\,a_1^2\,a_2\,a_4 + 4\,a_0\,a_2^2\,a_4 + 8\,a_0\,a_1\,a_3\,a_4 - 6\,a_0^2\,a_4^2 +\right.\\&\quad a_4^4 + 4\,a_1^3\,a_5 - 8\,a_0\,a_1\,a_2\,a_5 + 4\,a_0^2\,a_3\,a_5 - 4\,a_3\,a_4^2\,a_5 + 2\,a_3^2\,a_5^2 + 4\,a_2\,a_4\,a_5^2 - 4\,a_1\,a_5^3 -\\&\quad 4\,a_0\,a_1^2\,a_6 + 4\,a_0^2\,a_2\,a_6 + 4\,a_3^2\,a_4\,a_6 - 4\,a_2\,a_4^2\,a_6 - 8\,a_2\,a_3\,a_5\,a_6 + 8\,a_1\,a_4\,a_5\,a_6 +\\&\quad 4\,a_0\,a_5^2\,a_6 + 6\,a_2^2\,a_6^2 - 4\,a_1\,a_3\,a_6^2 - 4\,a_0\,a_4\,a_6^2 - a_6^4 + 4\,a_0^2\,a_1\,a_7 - 4\,a_3^3\,a_7 +\\&\quad 8\,a_2\,a_3\,a_4\,a_7 - 4\,a_1\,a_4^2\,a_7 - 4\,a_2^2\,a_5\,a_7 + 8\,a_1\,a_3\,a_5\,a_7 - 8\,a_0\,a_4\,a_5\,a_7 - 8\,a_1\,a_2\,a_6\,a_7 +\\&\quad \left.8\,a_0\,a_3\,a_6\,a_7 + 4\,a_5\,a_6^2\,a_7 + 2\,a_1^2\,a_7^2 + 4\,a_0\,a_2\,a_7^2 - 2\,a_5^2\,a_7^2 - 4\,a_4\,a_6\,a_7^2 + 4\,a_3\,a_7^3\right)^2 +\\&\left(a_1^4 - 4\,a_0\,a_1^2\,a_2 + 2\,a_0^2\,a_2^2 + 4\,a_0^2\,a_1\,a_3 - a_3^4 - 4\,a_0^3\,a_4 + 4\,a_2\,a_3^2\,a_4 - 2\,a_2^2\,a_4^2 - 4\,a_1\,a_3\,a_4^2 +\right.\\&\quad 4\,a_0\,a_4^3 - 4\,a_2^2\,a_3\,a_5 + 4\,a_1\,a_3^2\,a_5 + 8\,a_1\,a_2\,a_4\,a_5 - 8\,a_0\,a_3\,a_4\,a_5 - 6\,a_1^2\,a_5^2 + 4\,a_0\,a_2\,a_5^2 +\\&\quad a_5^4 + 4\,a_2^3\,a_6 - 8\,a_1\,a_2\,a_3\,a_6 + 4\,a_0\,a_3^2\,a_6 + 4\,a_1^2\,a_4\,a_6 - 8\,a_0\,a_2\,a_4\,a_6 + 8\,a_0\,a_1\,a_5\,a_6 -\\&\quad 4\,a_4\,a_5^2\,a_6 - 2\,a_0^2\,a_6^2 + 2\,a_4^2\,a_6^2 + 4\,a_3\,a_5\,a_6^2 - 4\,a_2\,a_6^3 - 4\,a_1\,a_2^2\,a_7 + 4\,a_1^2\,a_3\,a_7 +\\&\quad 8\,a_0\,a_2\,a_3\,a_7 - 8\,a_0\,a_1\,a_4\,a_7 - 4\,a_0^2\,a_5\,a_7 + 4\,a_4^2\,a_5\,a_7 - 4\,a_3\,a_5^2\,a_7 - 8\,a_3\,a_4\,a_6\,a_7 +\\&\quad \left.8\,a_2\,a_5\,a_6\,a_7 + 4\,a_1\,a_6^2\,a_7 + 6\,a_3^2\,a_7^2 - 4\,a_2\,a_4\,a_7^2 - 4\,a_1\,a_5\,a_7^2 - 4\,a_0\,a_6\,a_7^2 - a_7^4\right)^2\end{aligned}$$

It is a sum of squares of real numbers, so that we see ' norm3' is a nonnegative real number.

Norm expression when e = 4

Let φ be (φ^16 = -1, φ^32 = 1). We define "Replace" for it.

```
ruleφ1 = Flatten[Table[φ^(i+32 j) -> φ^i, {j, 1, 15}, {i, 0, 31}]];
```

```
First three and last three of it are
```

and are First it last of three^2

```
Take[ruleφ1, 3]
```

$$\{\varphi^{32} \to 1,\ \varphi^{33} \to \varphi,\ \varphi^{34} \to \varphi^2\}$$

```
Take[ruleφ1, -3]
```

$$\{\varphi^{509} \to \varphi^{29},\ \varphi^{510} \to \varphi^{30},\ \varphi^{511} \to \varphi^{31}\}$$

```
ruleφ2 = Table[φ^i → -φ^(i-16), {i, 16, 31}]
```

$$\begin{aligned}&\{\varphi^{16} \to -1,\ \varphi^{17} \to -\varphi,\ \varphi^{18} \to -\varphi^2,\ \varphi^{19} \to -\varphi^3,\ \varphi^{20} \to -\varphi^4,\ \varphi^{21} \to -\varphi^5,\ \varphi^{22} \to -\varphi^6,\ \varphi^{23} \to -\varphi^7,\\&\ \varphi^{24} \to -\varphi^8,\ \varphi^{25} \to -\varphi^9,\ \varphi^{26} \to -\varphi^{10},\ \varphi^{27} \to -\varphi^{11},\ \varphi^{28} \to -\varphi^{12},\ \varphi^{29} \to -\varphi^{13},\ \varphi^{30} \to -\varphi^{14},\ \varphi^{31} \to -\varphi^{15}\}\end{aligned}$$

Here, let the coefficients of polynomials be m[0],...,m[15].

```
s[r_] :=
  (a + b φ^r + c φ^(2 r) + d φ^(3 r) + e φ^(4 r) + f φ^(5 r) + g φ^(6 r) + h φ^(7 r) + j φ^(8 r) + k φ^(9 r) + m φ^(10 r) + n φ^(11 r) +
     p φ^(12 r) + q φ^(13 r) + s φ^(14 r) + t φ^(15 r)) /. ruleφ1 /. ruleφ2;
```

```
s[r_] := (Sum[a_i φ^(i r), {i, 0, 15}]) /. ruleφ1 /. ruleφ2;
```

```
s1 = s[1]
```

$$a_0 + \varphi\,a_1 + \varphi^2\,a_2 + \varphi^3\,a_3 + \varphi^4\,a_4 + \varphi^5\,a_5 + \varphi^6\,a_6 + \varphi^7\,a_7 + \varphi^8\,a_8 + \varphi^9\,a_9 + \varphi^{10}\,a_{10} + \varphi^{11}\,a_{11} + \varphi^{12}\,a_{12} + \varphi^{13}\,a_{13} + \varphi^{14}\,a_{14} + \varphi^{15}\,a_{15}$$

```
s3 = s[3]
```

$$a_0 + \varphi^3\,a_1 + \varphi^6\,a_2 + \varphi^9\,a_3 + \varphi^{12}\,a_4 + \varphi^{15}\,a_5 - \varphi^2\,a_6 - \varphi^5\,a_7 - \varphi^8\,a_8 - \varphi^{11}\,a_9 - \varphi^{14}\,a_{10} + \varphi\,a_{11} + \varphi^4\,a_{12} + \varphi^7\,a_{13} + \varphi^{10}\,a_{14} + \varphi^{13}\,a_{15}$$

We simplify their products by applying the rule $\varphi^2=\omega$ as same in the case of e ≤ 3.

```
ruleφω = Table[φ^(2 j) → ω^j, {j, 1, 7}]
```

$$\left\{\varphi^2 \to \omega,\ \varphi^4 \to \omega^2,\ \varphi^6 \to \omega^3,\ \varphi^8 \to \omega^4,\ \varphi^{10} \to \omega^5,\ \varphi^{12} \to \omega^6,\ \varphi^{14} \to \omega^7\right\}$$

```
prod1mod16 = Collect[Expand[MyProd[s[1], s[17]] /. ruleφω] /. ruleω1 /. ruleω2, ω]
```

$$\begin{aligned}
&a_0^2 - a_8^2 + 2 a_7 a_9 - 2 a_6 a_{10} + 2 a_5 a_{11} - 2 a_4 a_{12} + 2 a_3 a_{13} - 2 a_2 a_{14} + 2 a_1 a_{15} + \\
&\omega \left(-a_1^2 + 2 a_0 a_2 + a_9^2 - 2 a_8 a_{10} + 2 a_7 a_{11} - 2 a_6 a_{12} + 2 a_5 a_{13} - 2 a_4 a_{14} + 2 a_3 a_{15}\right) + \\
&\omega^2 \left(a_2^2 - 2 a_1 a_3 + 2 a_0 a_4 - a_{10}^2 + 2 a_9 a_{11} - 2 a_8 a_{12} + 2 a_7 a_{13} - 2 a_6 a_{14} + 2 a_5 a_{15}\right) + \\
&\omega^3 \left(-a_3^2 + 2 a_2 a_4 - 2 a_1 a_5 + 2 a_0 a_6 + a_{11}^2 - 2 a_{10} a_{12} + 2 a_9 a_{13} - 2 a_8 a_{14} + 2 a_7 a_{15}\right) + \\
&\omega^4 \left(a_4^2 - 2 a_3 a_5 + 2 a_2 a_6 - 2 a_1 a_7 + 2 a_0 a_8 - a_{12}^2 + 2 a_{11} a_{13} - 2 a_{10} a_{14} + 2 a_9 a_{15}\right) + \\
&\omega^5 \left(-a_5^2 + 2 a_4 a_6 - 2 a_3 a_7 + 2 a_2 a_8 - 2 a_1 a_9 + 2 a_0 a_{10} + a_{13}^2 - 2 a_{12} a_{14} + 2 a_{11} a_{15}\right) + \\
&\omega^6 \left(a_6^2 - 2 a_5 a_7 + 2 a_4 a_8 - 2 a_3 a_9 + 2 a_2 a_{10} - 2 a_1 a_{11} + 2 a_0 a_{12} - a_{14}^2 + 2 a_{13} a_{15}\right) + \\
&\omega^7 \left(-a_7^2 + 2 a_6 a_8 - 2 a_5 a_9 + 2 a_4 a_{10} - 2 a_3 a_{11} + 2 a_2 a_{12} - 2 a_1 a_{13} + 2 a_0 a_{14} + a_{15}^2\right)
\end{aligned}$$

Numbers of terms in the coefficients of this polynomial are :

```
Table[Length[Coefficient[prod1mod16, ω, i]], {i, 0, 7}]
```

$$\{9, 9, 9, 9, 9, 9, 9, 9\}$$

```
prod3mod16 = Collect[Expand[MyProd[s[3], s[19]] /. ruleφω] /. ruleω1 /. ruleω2, ω];
```

```
prod5mod16 = Collect[Expand[MyProd[s[5], s[21]] /. ruleφω] /. ruleω1 /. ruleω2, ω];
```

```
prod7mod16 = Collect[Expand[MyProd[s[7], s[23]] /. ruleφω] /. ruleω1 /. ruleω2, ω];
```

```
prod9mod16 = Collect[Expand[MyProd[s[9], s[25]] /. ruleφω] /. ruleω1 /. ruleω2, ω];
```

```
prod11mod16 =
  Collect[Expand[MyProd[s[11], s[27]] /. ruleφω] /. ruleω1 /. ruleω2, ω];
```

```
prod13mod16 =
  Collect[Expand[MyProd[s[13], s[29]] /. ruleφω] /. ruleω1 /. ruleω2, ω];
```

```
prod15mod16 =
  Collect[Expand[MyProd[s[15], s[31]] /. ruleφω] /. ruleω1 /. ruleω2, ω];
```

Each multiplication of two of these gives a third order polynomial of ζ :

```
prod1mod8 =
 Collect[Expand[MyProd[prod1mod16, prod9mod16] /. ruleωζ] /. ruleζ1 /. ruleζ2, ζ]
```

$$\begin{aligned}
&a_0^4 - a_4^4 + 4 a_3 a_4^2 a_5 - 2 a_3^2 a_5^2 - 4 a_2 a_4 a_5^2 + 4 a_1 a_5^3 - 4 a_3^2 a_4 a_6 + 4 a_2 a_4^2 a_6 + 8 a_2 a_3 a_5 a_6 - \\
&8 a_1 a_4 a_5 a_6 - 4 a_0 a_5^2 a_6 - 6 a_2^2 a_6^2 + 4 a_1 a_3 a_6^2 + 4 a_0 a_4 a_6^2 + 4 a_3^3 a_7 - 8 a_2 a_3 a_4 a_7 + \\
&4 a_1 a_4^2 a_7 + 4 a_2^2 a_5 a_7 - 8 a_1 a_3 a_5 a_7 + 8 a_0 a_4 a_5 a_7 + 8 a_1 a_2 a_6 a_7 - 8 a_0 a_3 a_6 a_7 - 2 a_1^2 a_7^2 - \\
&4 a_0 a_2 a_7^2 - 4 a_2 a_3^2 a_8 + 4 a_2^2 a_4 a_8 + 8 a_1 a_3 a_4 a_8 - 12 a_0 a_4^2 a_8 - 8 a_1 a_2 a_5 a_8 + 8 a_0 a_3 a_5 a_8 - \\
&4 a_1^2 a_6 a_8 + 8 a_0 a_2 a_6 a_8 + 8 a_0 a_1 a_7 a_8 - 6 a_0^2 a_8^2 + a_8^4 + 4 a_2^2 a_3 a_9 - 4 a_1 a_3^2 a_9 - 8 a_1 a_2 a_4 a_9 + \\
&8 a_0 a_3 a_4 a_9 + 12 a_1^2 a_5 a_9 - 8 a_0 a_2 a_5 a_9 - 8 a_0 a_1 a_6 a_9 + 4 a_0^2 a_7 a_9 - 4 a_7 a_8^2 a_9 + 2 a_7^2 a_9^2 + \\
&4 a_6 a_8 a_9^2 - 4 a_5 a_9^3 - 4 a_2^3 a_{10} + 8 a_1 a_2 a_3 a_{10} - 4 a_0 a_3^2 a_{10} - 4 a_1^2 a_4 a_{10} + 8 a_0 a_2 a_4 a_{10} - \\
&8 a_0 a_1 a_5 a_{10} + 4 a_0^2 a_6 a_{10} + 4 a_7^2 a_8 a_{10} - 4 a_6 a_8^2 a_{10} - 8 a_6 a_7 a_9 a_{10} + 8 a_5 a_8 a_9 a_{10} + \\
&4 a_4 a_9^2 a_{10} + 6 a_6^2 a_{10}^2 - 4 a_5 a_7 a_{10}^2 - 4 a_4 a_8 a_{10}^2 - 4 a_3 a_9 a_{10}^2 + 4 a_2 a_{10}^3 + 4 a_1 a_2^2 a_{11} - \\
&4 a_1^2 a_3 a_{11} - 8 a_0 a_2 a_3 a_{11} + 8 a_0 a_1 a_4 a_{11} + 4 a_0^2 a_5 a_{11} - 4 a_7^3 a_{11} + 8 a_6 a_7 a_8 a_{11} - 4 a_5 a_8^2 a_{11} - \\
&4 a_6^2 a_9 a_{11} + 8 a_5 a_7 a_9 a_{11} - 8 a_4 a_8 a_9 a_{11} + 4 a_3 a_9^2 a_{11} - 8 a_5 a_6 a_{10} a_{11} + 8 a_4 a_7 a_{10} a_{11} + \\
&8 a_3 a_8 a_{10} a_{11} - 8 a_2 a_9 a_{10} a_{11} - 4 a_1 a_{10}^2 a_{11} + 2 a_5^2 a_{11}^2 + 4 a_4 a_6 a_{11}^2 - 12 a_3 a_7 a_{11}^2 + \\
&4 a_2 a_8 a_{11}^2 + 4 a_1 a_9 a_{11}^2 + 4 a_0 a_{10} a_{11}^2 - 4 a_1^2 a_2 a_{12} + 4 a_0 a_2^2 a_{12} + 8 a_0 a_1 a_3 a_{12} - 12 a_0^2 a_4 a_{12} + \\
&4 a_6 a_7^2 a_{12} - 4 a_6^2 a_8 a_{12} - 8 a_5 a_7 a_8 a_{12} + 12 a_4 a_8^2 a_{12} + 8 a_5 a_6 a_9 a_{12} - 8 a_4 a_7 a_9 a_{12} - \\
&8 a_3 a_8 a_9 a_{12} + 4 a_2 a_9^2 a_{12} + 4 a_5^2 a_{10} a_{12} - 8 a_4 a_6 a_{10} a_{12} + 8 a_3 a_7 a_{10} a_{12} - 8 a_2 a_8 a_{10} a_{12} + \\
&8 a_1 a_9 a_{10} a_{12} - 4 a_0 a_{10}^2 a_{12} - 8 a_4 a_5 a_{11} a_{12} + 8 a_3 a_6 a_{11} a_{12} + 8 a_2 a_7 a_{11} a_{12} - 8 a_1 a_8 a_{11} a_{12} - \\
&8 a_0 a_9 a_{11} a_{12} + 6 a_4^2 a_{12}^2 - 4 a_3 a_5 a_{12}^2 - 4 a_2 a_6 a_{12}^2 - 4 a_1 a_7 a_{12}^2 + 12 a_0 a_8 a_{12}^2 - a_{12}^4 + 4 a_1^3 a_{13} - \\
&8 a_0 a_1 a_2 a_{13} + 4 a_0^2 a_3 a_{13} - 4 a_6^2 a_7 a_{13} + 4 a_5 a_7^2 a_{13} + 8 a_5 a_6 a_8 a_{13} - 8 a_4 a_7 a_8 a_{13} - \\
&4 a_3 a_8^2 a_{13} - 12 a_5^2 a_9 a_{13} + 8 a_4 a_6 a_9 a_{13} + 8 a_3 a_7 a_9 a_{13} + 8 a_2 a_8 a_9 a_{13} - 12 a_1 a_9^2 a_{13} +
\end{aligned}$$

$8 a_{4} a_{5} a_{10} a_{13} - 8 a_{3} a_{6} a_{10} a_{13} - 8 a_{2} a_{7} a_{10} a_{13} + 8 a_{1} a_{8} a_{10} a_{13} + 8 a_{0} a_{9} a_{10} a_{13} - 4 a_{4}^{2} a_{11} a_{13} +$
$8 a_{3} a_{5} a_{11} a_{13} - 8 a_{2} a_{6} a_{11} a_{13} + 8 a_{1} a_{7} a_{11} a_{13} - 8 a_{0} a_{8} a_{11} a_{13} - 8 a_{3} a_{4} a_{12} a_{13} +$
$8 a_{2} a_{5} a_{12} a_{13} + 8 a_{1} a_{6} a_{12} a_{13} - 8 a_{0} a_{7} a_{12} a_{13} + 4 a_{11} a_{12}^{2} a_{13} + 2 a_{3}^{2} a_{13}^{2} + 4 a_{2} a_{4} a_{13}^{2} -$
$12 a_{1} a_{5} a_{13}^{2} + 4 a_{0} a_{6} a_{13}^{2} - 2 a_{11}^{2} a_{13}^{2} - 4 a_{10} a_{12} a_{13}^{2} + 4 a_{9} a_{13}^{3} - 4 a_{0} a_{1}^{2} a_{14} + 4 a_{0}^{2} a_{2} a_{14} +$
$4 a_{6}^{3} a_{14} - 8 a_{5} a_{6} a_{7} a_{14} + 4 a_{4} a_{7}^{2} a_{14} + 4 a_{5}^{2} a_{8} a_{14} - 8 a_{4} a_{6} a_{8} a_{14} + 8 a_{3} a_{7} a_{8} a_{14} - 4 a_{2} a_{8}^{2} a_{14} +$
$8 a_{4} a_{5} a_{9} a_{14} - 8 a_{3} a_{6} a_{9} a_{14} - 8 a_{2} a_{7} a_{9} a_{14} + 8 a_{1} a_{8} a_{9} a_{14} + 4 a_{0} a_{9}^{2} a_{14} - 4 a_{4}^{2} a_{10} a_{14} -$
$8 a_{3} a_{5} a_{10} a_{14} + 24 a_{2} a_{6} a_{10} a_{14} - 8 a_{1} a_{7} a_{10} a_{14} - 8 a_{0} a_{8} a_{10} a_{14} + 8 a_{3} a_{4} a_{11} a_{14} -$
$8 a_{2} a_{5} a_{11} a_{14} - 8 a_{1} a_{6} a_{11} a_{14} + 8 a_{0} a_{7} a_{11} a_{14} + 4 a_{3}^{2} a_{12} a_{14} - 8 a_{2} a_{4} a_{12} a_{14} + 8 a_{1} a_{5} a_{12} a_{14} -$
$8 a_{0} a_{6} a_{12} a_{14} - 4 a_{11}^{2} a_{12} a_{14} + 4 a_{10} a_{12}^{2} a_{14} - 8 a_{2} a_{3} a_{13} a_{14} + 8 a_{1} a_{4} a_{13} a_{14} + 8 a_{0} a_{5} a_{13} a_{14} +$
$8 a_{10} a_{11} a_{13} a_{14} - 8 a_{9} a_{12} a_{13} a_{14} - 4 a_{8} a_{13}^{2} a_{14} + 6 a_{2}^{2} a_{14}^{2} - 4 a_{1} a_{3} a_{14}^{2} - 4 a_{0} a_{4} a_{14}^{2} - 6 a_{10}^{2} a_{14}^{2} +$
$4 a_{9} a_{11} a_{14}^{2} + 4 a_{8} a_{12} a_{14}^{2} + 4 a_{7} a_{13} a_{14}^{2} - 4 a_{6} a_{14}^{3} + 4 a_{0}^{2} a_{1} a_{15} - 4 a_{5} a_{6}^{2} a_{15} + 4 a_{5}^{2} a_{7} a_{15} +$
$8 a_{4} a_{6} a_{7} a_{15} - 12 a_{3} a_{7}^{2} a_{15} - 8 a_{4} a_{5} a_{8} a_{15} + 8 a_{3} a_{6} a_{8} a_{15} + 8 a_{2} a_{7} a_{8} a_{15} - 4 a_{1} a_{8}^{2} a_{15} -$
$4 a_{4}^{2} a_{9} a_{15} + 8 a_{3} a_{5} a_{9} a_{15} - 8 a_{2} a_{6} a_{9} a_{15} + 8 a_{1} a_{7} a_{9} a_{15} - 8 a_{0} a_{8} a_{9} a_{15} + 8 a_{3} a_{4} a_{10} a_{15} -$
$8 a_{2} a_{5} a_{10} a_{15} - 8 a_{1} a_{6} a_{10} a_{15} + 8 a_{0} a_{7} a_{10} a_{15} - 12 a_{3}^{2} a_{11} a_{15} + 8 a_{2} a_{4} a_{11} a_{15} + 8 a_{1} a_{5} a_{11} a_{15} +$
$8 a_{0} a_{6} a_{11} a_{15} + 4 a_{11}^{3} a_{15} + 8 a_{2} a_{3} a_{12} a_{15} - 8 a_{1} a_{4} a_{12} a_{15} - 8 a_{0} a_{5} a_{12} a_{15} - 8 a_{10} a_{11} a_{12} a_{15} +$
$4 a_{9} a_{12}^{2} a_{15} - 4 a_{2}^{2} a_{13} a_{15} + 8 a_{1} a_{3} a_{13} a_{15} - 8 a_{0} a_{4} a_{13} a_{15} + 4 a_{10}^{2} a_{13} a_{15} - 8 a_{9} a_{11} a_{13} a_{15} +$
$8 a_{8} a_{12} a_{13} a_{15} - 4 a_{7} a_{13}^{2} a_{15} - 8 a_{1} a_{2} a_{14} a_{15} + 8 a_{0} a_{3} a_{14} a_{15} + 8 a_{9} a_{10} a_{14} a_{15} -$
$8 a_{8} a_{11} a_{14} a_{15} - 8 a_{7} a_{12} a_{14} a_{15} + 8 a_{6} a_{13} a_{14} a_{15} + 4 a_{5} a_{14}^{2} a_{15} + 2 a_{1}^{2} a_{15}^{2} + 4 a_{0} a_{2} a_{15}^{2} -$
$2 a_{9}^{2} a_{15}^{2} - 4 a_{8} a_{10} a_{15}^{2} + 12 a_{7} a_{11} a_{15}^{2} - 4 a_{6} a_{12} a_{15}^{2} - 4 a_{5} a_{13} a_{15}^{2} - 4 a_{4} a_{14} a_{15}^{2} + 4 a_{3} a_{15}^{3} +$
$\zeta \left(-a_{1}^{4} + 4 a_{0} a_{1}^{2} a_{2} - 2 a_{0}^{2} a_{2}^{2} - 4 a_{0}^{2} a_{1} a_{3} + 4 a_{0}^{3} a_{4} + a_{5}^{4} - 4 a_{4} a_{5}^{2} a_{6} + 2 a_{4}^{2} a_{6}^{2} + 4 a_{3} a_{5} a_{6}^{2} - 4 a_{2} a_{6}^{3} +\right.$
$4 a_{4}^{2} a_{5} a_{7} - 4 a_{3} a_{5}^{2} a_{7} - 8 a_{3} a_{4} a_{6} a_{7} + 8 a_{2} a_{5} a_{6} a_{7} + 4 a_{1} a_{6}^{2} a_{7} + 6 a_{3}^{2} a_{7}^{2} - 4 a_{2} a_{4} a_{7}^{2} -$
$4 a_{1} a_{5} a_{7}^{2} - 4 a_{0} a_{6} a_{7}^{2} - 4 a_{4}^{3} a_{8} + 8 a_{3} a_{4} a_{5} a_{8} - 4 a_{2} a_{5}^{2} a_{8} - 4 a_{3}^{2} a_{6} a_{8} + 8 a_{2} a_{4} a_{6} a_{8} -$
$8 a_{1} a_{5} a_{6} a_{8} + 4 a_{0} a_{6}^{2} a_{8} - 8 a_{2} a_{3} a_{7} a_{8} + 8 a_{1} a_{4} a_{7} a_{8} + 8 a_{0} a_{5} a_{7} a_{8} + 2 a_{2}^{2} a_{8}^{2} + 4 a_{1} a_{3} a_{8}^{2} -$
$12 a_{0} a_{4} a_{8}^{2} + 4 a_{3} a_{4}^{2} a_{9} - 4 a_{3}^{2} a_{5} a_{9} - 8 a_{2} a_{4} a_{5} a_{9} + 12 a_{1} a_{5}^{2} a_{9} + 8 a_{2} a_{3} a_{6} a_{9} - 8 a_{1} a_{4} a_{6} a_{9} -$
$8 a_{0} a_{5} a_{6} a_{9} + 4 a_{2}^{2} a_{7} a_{9} - 8 a_{1} a_{3} a_{7} a_{9} + 8 a_{0} a_{4} a_{7} a_{9} - 8 a_{1} a_{2} a_{8} a_{9} + 8 a_{0} a_{3} a_{8} a_{9} +$
$6 a_{1}^{2} a_{9}^{2} - 4 a_{0} a_{2} a_{9}^{2} - a_{9}^{4} - 4 a_{3}^{2} a_{4} a_{10} + 4 a_{2} a_{4}^{2} a_{10} + 8 a_{2} a_{3} a_{5} a_{10} - 8 a_{1} a_{4} a_{5} a_{10} -$
$4 a_{0} a_{5}^{2} a_{10} - 12 a_{2}^{2} a_{6} a_{10} + 8 a_{1} a_{3} a_{6} a_{10} + 8 a_{0} a_{4} a_{6} a_{10} + 8 a_{1} a_{2} a_{7} a_{10} - 8 a_{0} a_{3} a_{7} a_{10} -$
$4 a_{1}^{2} a_{8} a_{10} + 8 a_{0} a_{2} a_{8} a_{10} - 8 a_{0} a_{1} a_{9} a_{10} + 4 a_{8} a_{9}^{2} a_{10} + 2 a_{0}^{2} a_{10}^{2} - 2 a_{8}^{2} a_{10}^{2} - 4 a_{7} a_{9} a_{10}^{2} +$
$4 a_{6} a_{10}^{3} + 4 a_{3}^{3} a_{11} - 8 a_{2} a_{3} a_{4} a_{11} + 4 a_{1} a_{4}^{2} a_{11} + 4 a_{2}^{2} a_{5} a_{11} - 8 a_{1} a_{3} a_{5} a_{11} + 8 a_{0} a_{4} a_{5} a_{11} +$
$8 a_{1} a_{2} a_{6} a_{11} - 8 a_{0} a_{3} a_{6} a_{11} - 4 a_{1}^{2} a_{7} a_{11} - 8 a_{0} a_{2} a_{7} a_{11} + 8 a_{0} a_{1} a_{8} a_{11} + 4 a_{0}^{2} a_{9} a_{11} -$
$4 a_{8}^{2} a_{9} a_{11} + 4 a_{7} a_{9}^{2} a_{11} + 8 a_{7} a_{8} a_{10} a_{11} - 8 a_{6} a_{9} a_{10} a_{11} - 4 a_{5} a_{10}^{2} a_{11} - 6 a_{7}^{2} a_{11}^{2} + 4 a_{6} a_{8} a_{11}^{2} +$
$4 a_{5} a_{9} a_{11}^{2} + 4 a_{4} a_{10} a_{11}^{2} - 4 a_{3} a_{11}^{3} - 4 a_{2} a_{3}^{2} a_{12} + 4 a_{2}^{2} a_{4} a_{12} + 8 a_{1} a_{3} a_{4} a_{12} - 12 a_{0} a_{4}^{2} a_{12} -$
$8 a_{1} a_{2} a_{5} a_{12} + 8 a_{0} a_{3} a_{5} a_{12} - 4 a_{1}^{2} a_{6} a_{12} + 8 a_{0} a_{2} a_{6} a_{12} + 8 a_{0} a_{1} a_{7} a_{12} - 12 a_{0}^{2} a_{8} a_{12} +$
$4 a_{8}^{3} a_{12} - 8 a_{7} a_{8} a_{9} a_{12} + 4 a_{6} a_{9}^{2} a_{12} + 4 a_{7}^{2} a_{10} a_{12} - 8 a_{6} a_{8} a_{10} a_{12} + 8 a_{5} a_{9} a_{10} a_{12} -$
$4 a_{4} a_{10}^{2} a_{12} + 8 a_{6} a_{7} a_{11} a_{12} - 8 a_{5} a_{8} a_{11} a_{12} - 8 a_{4} a_{9} a_{11} a_{12} + 8 a_{3} a_{10} a_{11} a_{12} + 4 a_{2} a_{11}^{2} a_{12} -$
$2 a_{6}^{2} a_{12}^{2} - 4 a_{5} a_{7} a_{12}^{2} + 12 a_{4} a_{8} a_{12}^{2} - 4 a_{3} a_{9} a_{12}^{2} - 4 a_{2} a_{10} a_{12}^{2} - 4 a_{1} a_{11} a_{12}^{2} + 4 a_{0} a_{12}^{3} +$
$4 a_{2}^{2} a_{3} a_{13} - 4 a_{1} a_{3}^{2} a_{13} - 8 a_{1} a_{2} a_{4} a_{13} + 8 a_{0} a_{3} a_{4} a_{13} + 12 a_{1}^{2} a_{5} a_{13} - 8 a_{0} a_{2} a_{5} a_{13} -$
$8 a_{0} a_{1} a_{6} a_{13} + 4 a_{0}^{2} a_{7} a_{13} - 4 a_{7} a_{8}^{2} a_{13} + 4 a_{7}^{2} a_{9} a_{13} + 8 a_{6} a_{8} a_{9} a_{13} - 12 a_{5} a_{9}^{2} a_{13} -$
$8 a_{6} a_{7} a_{10} a_{13} + 8 a_{5} a_{8} a_{10} a_{13} + 8 a_{4} a_{9} a_{10} a_{13} - 4 a_{3} a_{10}^{2} a_{13} - 4 a_{6}^{2} a_{11} a_{13} + 8 a_{5} a_{7} a_{11} a_{13} -$
$8 a_{4} a_{8} a_{11} a_{13} + 8 a_{3} a_{9} a_{11} a_{13} - 8 a_{2} a_{10} a_{11} a_{13} + 4 a_{1} a_{11}^{2} a_{13} + 8 a_{5} a_{6} a_{12} a_{13} -$
$8 a_{4} a_{7} a_{12} a_{13} - 8 a_{3} a_{8} a_{12} a_{13} + 8 a_{2} a_{9} a_{12} a_{13} + 8 a_{1} a_{10} a_{12} a_{13} - 8 a_{0} a_{11} a_{12} a_{13} -$
$6 a_{5}^{2} a_{13}^{2} + 4 a_{4} a_{6} a_{13}^{2} + 4 a_{3} a_{7} a_{13}^{2} + 4 a_{2} a_{8} a_{13}^{2} - 12 a_{1} a_{9} a_{13}^{2} + 4 a_{0} a_{10} a_{13}^{2} + a_{13}^{4} - 4 a_{2}^{3} a_{14} +$
$8 a_{1} a_{2} a_{3} a_{14} - 4 a_{0} a_{3}^{2} a_{14} - 4 a_{1}^{2} a_{4} a_{14} + 8 a_{0} a_{2} a_{4} a_{14} - 8 a_{0} a_{1} a_{5} a_{14} + 4 a_{0}^{2} a_{6} a_{14} +$
$4 a_{7}^{2} a_{8} a_{14} - 4 a_{6} a_{8}^{2} a_{14} - 8 a_{6} a_{7} a_{9} a_{14} + 8 a_{5} a_{8} a_{9} a_{14} + 4 a_{4} a_{9}^{2} a_{14} + 12 a_{6}^{2} a_{10} a_{14} -$
$8 a_{5} a_{7} a_{10} a_{14} - 8 a_{4} a_{8} a_{10} a_{14} - 8 a_{3} a_{9} a_{10} a_{14} + 12 a_{2} a_{10}^{2} a_{14} - 8 a_{5} a_{6} a_{11} a_{14} +$
$8 a_{4} a_{7} a_{11} a_{14} + 8 a_{3} a_{8} a_{11} a_{14} - 8 a_{2} a_{9} a_{11} a_{14} - 8 a_{1} a_{10} a_{11} a_{14} + 4 a_{0} a_{11}^{2} a_{14} + 4 a_{5}^{2} a_{12} a_{14} -$
$8 a_{4} a_{6} a_{12} a_{14} + 8 a_{3} a_{7} a_{12} a_{14} - 8 a_{2} a_{8} a_{12} a_{14} + 8 a_{1} a_{9} a_{12} a_{14} - 8 a_{0} a_{10} a_{12} a_{14} +$
$8 a_{4} a_{5} a_{13} a_{14} - 8 a_{3} a_{6} a_{13} a_{14} - 8 a_{2} a_{7} a_{13} a_{14} + 8 a_{1} a_{8} a_{13} a_{14} + 8 a_{0} a_{9} a_{13} a_{14} -$
$4 a_{12} a_{13}^{2} a_{14} - 2 a_{4}^{2} a_{14}^{2} - 4 a_{3} a_{5} a_{14}^{2} + 12 a_{2} a_{6} a_{14}^{2} - 4 a_{1} a_{7} a_{14}^{2} - 4 a_{0} a_{8} a_{14}^{2} + 2 a_{12}^{2} a_{14}^{2} +$
$4 a_{11} a_{13} a_{14}^{2} - 4 a_{10} a_{14}^{3} + 4 a_{1} a_{2}^{2} a_{15} - 4 a_{1}^{2} a_{3} a_{15} - 8 a_{0} a_{2} a_{3} a_{15} + 8 a_{0} a_{1} a_{4} a_{15} + 4 a_{0}^{2} a_{5} a_{15} -$
$4 a_{7}^{3} a_{15} + 8 a_{6} a_{7} a_{8} a_{15} - 4 a_{5} a_{8}^{2} a_{15} - 4 a_{6}^{2} a_{9} a_{15} + 8 a_{5} a_{7} a_{9} a_{15} - 8 a_{4} a_{8} a_{9} a_{15} +$
$4 a_{3} a_{9}^{2} a_{15} - 8 a_{5} a_{6} a_{10} a_{15} + 8 a_{4} a_{7} a_{10} a_{15} + 8 a_{3} a_{8} a_{10} a_{15} - 8 a_{2} a_{9} a_{10} a_{15} - 4 a_{1} a_{10}^{2} a_{15} +$
$4 a_{5}^{2} a_{11} a_{15} + 8 a_{4} a_{6} a_{11} a_{15} - 24 a_{3} a_{7} a_{11} a_{15} + 8 a_{2} a_{8} a_{11} a_{15} + 8 a_{1} a_{9} a_{11} a_{15} +$
$8 a_{0} a_{10} a_{11} a_{15} - 8 a_{4} a_{5} a_{12} a_{15} + 8 a_{3} a_{6} a_{12} a_{15} + 8 a_{2} a_{7} a_{12} a_{15} - 8 a_{1} a_{8} a_{12} a_{15} -$
$8 a_{0} a_{9} a_{12} a_{15} - 4 a_{4}^{2} a_{13} a_{15} + 8 a_{3} a_{5} a_{13} a_{15} - 8 a_{2} a_{6} a_{13} a_{15} + 8 a_{1} a_{7} a_{13} a_{15} -$
$8 a_{0} a_{8} a_{13} a_{15} + 4 a_{12}^{2} a_{13} a_{15} - 4 a_{11} a_{13}^{2} a_{15} + 8 a_{3} a_{4} a_{14} a_{15} - 8 a_{2} a_{5} a_{14} a_{15} - 8 a_{1} a_{6} a_{14} a_{15} +$

$$8 a_0 a_7 a_{14} a_{15} - 8 a_{11} a_{12} a_{14} a_{15} + 8 a_{10} a_{13} a_{14} a_{15} + 4 a_9 a_{14}^2 a_{15} - 6 a_3^2 a_{15}^2 + 4 a_2 a_4 a_{15}^2 +$$
$$4 a_1 a_5 a_{15}^2 + 4 a_0 a_6 a_{15}^2 + 6 a_{11}^2 a_{15}^2 - 4 a_{10} a_{12} a_{15}^2 - 4 a_9 a_{13} a_{15}^2 - 4 a_8 a_{14} a_{15}^2 + 4 a_7 a_{15}^3 \Big) +$$
$$\zeta^2 \Big(a_2^4 - 4 a_1 a_2^2 a_3 + 2 a_1^2 a_3^2 + 4 a_0 a_2 a_3^2 + 4 a_1^2 a_2 a_4 - 4 a_0 a_2^2 a_4 - 8 a_0 a_1 a_3 a_4 + 6 a_0^2 a_4^2 -$$
$$4 a_1^3 a_5 + 8 a_0 a_1 a_2 a_5 - 4 a_0^2 a_3 a_5 + 4 a_0 a_1^2 a_6 - 4 a_0^2 a_2 a_6 - a_6^4 - 4 a_0^2 a_1 a_7 + 4 a_5 a_6^2 a_7 -$$
$$2 a_5^2 a_7^2 - 4 a_4 a_6 a_7^2 + 4 a_3 a_7^3 + 4 a_0^3 a_8 - 4 a_5^2 a_6 a_8 + 4 a_4 a_6^2 a_8 + 8 a_4 a_5 a_7 a_8 - 8 a_3 a_6 a_7 a_8 -$$
$$4 a_2 a_7^2 a_8 - 6 a_4^2 a_8^2 + 4 a_3 a_5 a_8^2 + 4 a_2 a_6 a_8^2 + 4 a_1 a_7 a_8^2 - 4 a_0 a_8^3 + 4 a_5^3 a_9 - 8 a_4 a_5 a_6 a_9 +$$
$$4 a_3 a_6^2 a_9 + 4 a_4^2 a_7 a_9 - 8 a_3 a_5 a_7 a_9 + 8 a_2 a_6 a_7 a_9 - 4 a_1 a_7^2 a_9 + 8 a_3 a_4 a_8 a_9 - 8 a_2 a_5 a_8 a_9 -$$
$$8 a_1 a_6 a_8 a_9 + 8 a_0 a_7 a_8 a_9 - 2 a_3^2 a_9^2 - 4 a_2 a_4 a_9^2 + 12 a_1 a_5 a_9^2 - 4 a_0 a_6 a_9^2 - 4 a_4 a_5^2 a_{10} +$$
$$4 a_4^2 a_6 a_{10} + 8 a_3 a_5 a_6 a_{10} - 12 a_2 a_6^2 a_{10} - 8 a_3 a_4 a_7 a_{10} + 8 a_2 a_5 a_7 a_{10} + 8 a_1 a_6 a_7 a_{10} -$$
$$4 a_0 a_7^2 a_{10} - 4 a_3^2 a_8 a_{10} + 8 a_2 a_4 a_8 a_{10} - 8 a_1 a_5 a_8 a_{10} + 8 a_0 a_6 a_8 a_{10} + 8 a_2 a_3 a_9 a_{10} -$$
$$8 a_1 a_4 a_9 a_{10} - 8 a_0 a_5 a_9 a_{10} - 6 a_2^2 a_{10}^2 + 4 a_1 a_3 a_{10}^2 + 4 a_0 a_4 a_{10}^2 + a_{10}^4 + 4 a_4^2 a_5 a_{11} -$$
$$4 a_3 a_5^2 a_{11} - 8 a_3 a_4 a_6 a_{11} + 8 a_2 a_5 a_6 a_{11} + 4 a_1 a_6^2 a_{11} + 12 a_3^2 a_7 a_{11} - 8 a_2 a_4 a_7 a_{11} -$$
$$8 a_1 a_5 a_7 a_{11} - 8 a_0 a_6 a_7 a_{11} - 8 a_2 a_3 a_8 a_{11} + 8 a_1 a_4 a_8 a_{11} + 8 a_0 a_5 a_8 a_{11} + 4 a_2^2 a_9 a_{11} -$$
$$8 a_1 a_3 a_9 a_{11} + 8 a_0 a_4 a_9 a_{11} + 8 a_1 a_2 a_{10} a_{11} - 8 a_0 a_3 a_{10} a_{11} - 4 a_9 a_{10}^2 a_{11} - 2 a_1^2 a_{11}^2 -$$
$$4 a_0 a_2 a_{11}^2 + 2 a_9^2 a_{11}^2 + 4 a_8 a_{10} a_{11}^2 - 4 a_7 a_{11}^3 - 4 a_4^3 a_{12} + 8 a_3 a_4 a_5 a_{12} - 4 a_2 a_5^2 a_{12} -$$
$$4 a_3^2 a_6 a_{12} + 8 a_2 a_4 a_6 a_{12} - 8 a_1 a_5 a_6 a_{12} + 4 a_0 a_6^2 a_{12} - 8 a_2 a_3 a_7 a_{12} + 8 a_1 a_4 a_7 a_{12} +$$
$$8 a_0 a_5 a_7 a_{12} + 4 a_2^2 a_8 a_{12} + 8 a_1 a_3 a_8 a_{12} - 24 a_0 a_4 a_8 a_{12} - 8 a_1 a_2 a_9 a_{12} + 8 a_0 a_3 a_9 a_{12} -$$
$$4 a_1^2 a_{10} a_{12} + 8 a_0 a_2 a_{10} a_{12} + 4 a_9^2 a_{10} a_{12} - 4 a_8 a_{10}^2 a_{12} + 8 a_0 a_1 a_{11} a_{12} - 8 a_8 a_9 a_{11} a_{12} +$$
$$8 a_7 a_{10} a_{11} a_{12} + 4 a_6 a_{11}^2 a_{12} - 6 a_0^2 a_{12}^2 + 6 a_8^2 a_{12}^2 - 4 a_7 a_9 a_{12}^2 - 4 a_6 a_{10} a_{12}^2 - 4 a_5 a_{11} a_{12}^2 +$$
$$4 a_4 a_{12}^3 + 4 a_3 a_4^2 a_{13} - 4 a_3^2 a_5 a_{13} - 8 a_2 a_4 a_5 a_{13} + 12 a_1 a_5^2 a_{13} + 8 a_2 a_3 a_6 a_{13} -$$
$$8 a_1 a_4 a_6 a_{13} - 8 a_0 a_5 a_6 a_{13} + 4 a_2^2 a_7 a_{13} - 8 a_1 a_3 a_7 a_{13} + 8 a_0 a_4 a_7 a_{13} - 8 a_1 a_2 a_8 a_{13} +$$
$$8 a_0 a_3 a_8 a_{13} + 12 a_1^2 a_9 a_{13} - 8 a_0 a_2 a_9 a_{13} - 4 a_9^3 a_{13} - 8 a_0 a_1 a_{10} a_{13} + 8 a_8 a_9 a_{10} a_{13} -$$
$$4 a_7 a_{10}^2 a_{13} + 4 a_0^2 a_{11} a_{13} - 4 a_8^2 a_{11} a_{13} + 8 a_7 a_9 a_{11} a_{13} - 8 a_6 a_{10} a_{11} a_{13} + 4 a_5 a_{11}^2 a_{13} -$$
$$8 a_7 a_8 a_{12} a_{13} + 8 a_6 a_9 a_{12} a_{13} + 8 a_5 a_{10} a_{12} a_{13} - 8 a_4 a_{11} a_{12} a_{13} - 4 a_3 a_{12}^2 a_{13} + 2 a_7^2 a_{13}^2 +$$
$$4 a_6 a_8 a_{13}^2 - 12 a_5 a_9 a_{13}^2 + 4 a_4 a_{10} a_{13}^2 + 4 a_3 a_{11} a_{13}^2 + 4 a_2 a_{12} a_{13}^2 - 4 a_1 a_{13}^3 - 4 a_3^2 a_4 a_{14} +$$
$$4 a_2 a_4^2 a_{14} + 8 a_2 a_3 a_5 a_{14} - 8 a_1 a_4 a_5 a_{14} - 4 a_0 a_5^2 a_{14} - 12 a_2^2 a_6 a_{14} + 8 a_1 a_3 a_6 a_{14} +$$
$$8 a_0 a_4 a_6 a_{14} + 8 a_1 a_2 a_7 a_{14} - 8 a_0 a_3 a_7 a_{14} - 4 a_1^2 a_8 a_{14} + 8 a_0 a_2 a_8 a_{14} - 8 a_0 a_1 a_9 a_{14} +$$
$$4 a_8 a_9^2 a_{14} + 4 a_0^2 a_{10} a_{14} - 4 a_8^2 a_{10} a_{14} - 8 a_7 a_9 a_{10} a_{14} + 12 a_6 a_{10}^2 a_{14} + 8 a_7 a_8 a_{11} a_{14} -$$
$$8 a_6 a_9 a_{11} a_{14} - 8 a_5 a_{10} a_{11} a_{14} + 4 a_4 a_{11}^2 a_{14} + 4 a_7^2 a_{12} a_{14} - 8 a_6 a_8 a_{12} a_{14} + 8 a_5 a_9 a_{12} a_{14} -$$
$$8 a_4 a_{10} a_{12} a_{14} + 8 a_3 a_{11} a_{12} a_{14} - 4 a_2 a_{12}^2 a_{14} - 8 a_6 a_7 a_{13} a_{14} + 8 a_5 a_8 a_{13} a_{14} +$$
$$8 a_4 a_9 a_{13} a_{14} - 8 a_3 a_{10} a_{13} a_{14} - 8 a_2 a_{11} a_{13} a_{14} + 8 a_1 a_{12} a_{13} a_{14} + 4 a_0 a_{13}^2 a_{14} +$$
$$6 a_6^2 a_{14}^2 - 4 a_5 a_7 a_{14}^2 - 4 a_4 a_8 a_{14}^2 - 4 a_3 a_9 a_{14}^2 + 12 a_2 a_{10} a_{14}^2 - 4 a_1 a_{11} a_{14}^2 - 4 a_0 a_{12} a_{14}^2 -$$
$$a_{14}^4 + 4 a_3^3 a_{15} - 8 a_2 a_3 a_4 a_{15} + 4 a_1 a_4^2 a_{15} + 4 a_2^2 a_5 a_{15} - 8 a_1 a_3 a_5 a_{15} + 8 a_0 a_4 a_5 a_{15} +$$
$$8 a_1 a_2 a_6 a_{15} - 8 a_0 a_3 a_6 a_{15} - 4 a_1^2 a_7 a_{15} - 8 a_0 a_2 a_7 a_{15} + 8 a_0 a_1 a_8 a_{15} + 4 a_0^2 a_9 a_{15} -$$
$$4 a_8^2 a_9 a_{15} + 4 a_7 a_9^2 a_{15} + 8 a_7 a_8 a_{10} a_{15} - 8 a_6 a_9 a_{10} a_{15} - 4 a_5 a_{10}^2 a_{15} - 12 a_7^2 a_{11} a_{15} +$$
$$8 a_6 a_8 a_{11} a_{15} + 8 a_5 a_9 a_{11} a_{15} + 8 a_4 a_{10} a_{11} a_{15} - 12 a_3 a_{11}^2 a_{15} + 8 a_6 a_7 a_{12} a_{15} -$$
$$8 a_5 a_8 a_{12} a_{15} - 8 a_4 a_9 a_{12} a_{15} + 8 a_3 a_{10} a_{12} a_{15} + 8 a_2 a_{11} a_{12} a_{15} - 4 a_1 a_{12}^2 a_{15} -$$
$$4 a_6^2 a_{13} a_{15} + 8 a_5 a_7 a_{13} a_{15} - 8 a_4 a_8 a_{13} a_{15} + 8 a_3 a_9 a_{13} a_{15} - 8 a_2 a_{10} a_{13} a_{15} +$$
$$8 a_1 a_{11} a_{13} a_{15} - 8 a_0 a_{12} a_{13} a_{15} - 8 a_5 a_6 a_{14} a_{15} + 8 a_4 a_7 a_{14} a_{15} + 8 a_3 a_8 a_{14} a_{15} -$$
$$8 a_2 a_9 a_{14} a_{15} - 8 a_1 a_{10} a_{14} a_{15} + 8 a_0 a_{11} a_{14} a_{15} + 4 a_{13} a_{14}^2 a_{15} + 2 a_5^2 a_{15}^2 + 4 a_4 a_6 a_{15}^2 -$$
$$12 a_3 a_7 a_{15}^2 + 4 a_2 a_8 a_{15}^2 + 4 a_1 a_9 a_{15}^2 + 4 a_0 a_{10} a_{15}^2 - 2 a_{13}^2 a_{15}^2 - 4 a_{12} a_{14} a_{15}^2 + 4 a_{11} a_{15}^3 \Big) +$$
$$\zeta^3 \Big(-a_3^4 + 4 a_2 a_3^2 a_4 - 2 a_2^2 a_4^2 - 4 a_1 a_3 a_4^2 + 4 a_0 a_4^3 - 4 a_2^2 a_3 a_5 + 4 a_1 a_3^2 a_5 + 8 a_1 a_2 a_4 a_5 -$$
$$8 a_0 a_3 a_4 a_5 - 6 a_1^2 a_5^2 + 4 a_0 a_2 a_5^2 + 4 a_2^3 a_6 - 8 a_1 a_2 a_3 a_6 + 4 a_0 a_3^2 a_6 + 4 a_1^2 a_4 a_6 -$$
$$8 a_0 a_2 a_4 a_6 + 8 a_0 a_1 a_5 a_6 - 2 a_0^2 a_6^2 - 4 a_1 a_2^2 a_7 + 4 a_1^2 a_3 a_7 + 8 a_0 a_2 a_3 a_7 - 8 a_0 a_1 a_4 a_7 -$$
$$4 a_0^2 a_5 a_7 + a_7^4 + 4 a_1^2 a_2 a_8 - 4 a_0 a_2^2 a_8 - 8 a_0 a_1 a_3 a_8 + 12 a_0^2 a_4 a_8 - 4 a_6 a_7^2 a_8 + 2 a_6^2 a_8^2 +$$
$$4 a_5 a_7 a_8^2 - 4 a_4 a_8^3 - 4 a_1^3 a_9 + 8 a_0 a_1 a_2 a_9 - 4 a_0^2 a_3 a_9 + 4 a_6^2 a_7 a_9 - 4 a_5 a_7^2 a_9 - 8 a_5 a_6 a_8 a_9 +$$
$$8 a_4 a_7 a_8 a_9 + 4 a_3 a_8^2 a_9 + 6 a_5^2 a_9^2 - 4 a_4 a_6 a_9^2 - 4 a_3 a_7 a_9^2 - 4 a_2 a_8 a_9^2 + 4 a_1 a_9^3 + 4 a_0 a_1^2 a_{10} -$$
$$4 a_0^2 a_2 a_{10} - 4 a_6^3 a_{10} + 8 a_5 a_6 a_7 a_{10} - 4 a_4 a_7^2 a_{10} - 4 a_5^2 a_8 a_{10} + 8 a_4 a_6 a_8 a_{10} - 8 a_3 a_7 a_8 a_{10} +$$
$$4 a_2 a_8^2 a_{10} - 8 a_4 a_5 a_9 a_{10} + 8 a_3 a_6 a_9 a_{10} + 8 a_2 a_7 a_9 a_{10} - 8 a_1 a_8 a_9 a_{10} - 4 a_0 a_9^2 a_{10} +$$
$$2 a_4^2 a_{10}^2 + 4 a_3 a_5 a_{10}^2 - 12 a_2 a_6 a_{10}^2 + 4 a_1 a_7 a_{10}^2 + 4 a_0 a_8 a_{10}^2 - 4 a_0^2 a_1 a_{11} + 4 a_5 a_6^2 a_{11} -$$
$$4 a_5^2 a_7 a_{11} - 8 a_4 a_6 a_7 a_{11} + 12 a_3 a_7^2 a_{11} + 8 a_4 a_5 a_8 a_{11} - 8 a_3 a_6 a_8 a_{11} - 8 a_2 a_7 a_8 a_{11} +$$
$$4 a_1 a_8^2 a_{11} + 4 a_4^2 a_9 a_{11} - 8 a_3 a_5 a_9 a_{11} + 8 a_2 a_6 a_9 a_{11} - 8 a_1 a_7 a_9 a_{11} + 8 a_0 a_8 a_9 a_{11} -$$
$$8 a_3 a_4 a_{10} a_{11} + 8 a_2 a_5 a_{10} a_{11} + 8 a_1 a_6 a_{10} a_{11} - 8 a_0 a_7 a_{10} a_{11} + 6 a_3^2 a_{11}^2 - 4 a_2 a_4 a_{11}^2 -$$
$$4 a_1 a_5 a_{11}^2 - 4 a_0 a_6 a_{11}^2 - a_{11}^4 + 4 a_0^3 a_{12} - 4 a_5^2 a_6 a_{12} + 4 a_4 a_6^2 a_{12} + 8 a_4 a_5 a_7 a_{12} -$$
$$8 a_3 a_6 a_7 a_{12} - 4 a_2 a_7^2 a_{12} - 12 a_4^2 a_8 a_{12} + 8 a_3 a_5 a_8 a_{12} + 8 a_2 a_6 a_8 a_{12} + 8 a_1 a_7 a_8 a_{12} -$$
$$12 a_0 a_8^2 a_{12} + 8 a_3 a_4 a_9 a_{12} - 8 a_2 a_5 a_9 a_{12} - 8 a_1 a_6 a_9 a_{12} + 8 a_0 a_7 a_9 a_{12} - 4 a_3^2 a_{10} a_{12} +$$

$$
\begin{aligned}
&8\,a_2\,a_4\,a_{10}\,a_{12} - 8\,a_1\,a_5\,a_{10}\,a_{12} + 8\,a_0\,a_6\,a_{10}\,a_{12} - 8\,a_2\,a_3\,a_{11}\,a_{12} + 8\,a_1\,a_4\,a_{11}\,a_{12} +\\
&8\,a_0\,a_5\,a_{11}\,a_{12} + 4\,a_{10}\,a_{11}^2\,a_{12} + 2\,a_2^2\,a_{12}^2 + 4\,a_1\,a_3\,a_{12}^2 - 12\,a_0\,a_4\,a_{12}^2 - 2\,a_{10}^2\,a_{12}^2 - 4\,a_9\,a_{11}\,a_{12}^2 +\\
&4\,a_8\,a_{12}^3 + 4\,a_5^3\,a_{13} - 8\,a_4\,a_5\,a_6\,a_{13} + 4\,a_3\,a_6^2\,a_{13} + 4\,a_4^2\,a_7\,a_{13} - 8\,a_3\,a_5\,a_7\,a_{13} + 8\,a_2\,a_6\,a_7\,a_{13} -\\
&4\,a_1\,a_7^2\,a_{13} + 8\,a_3\,a_4\,a_8\,a_{13} - 8\,a_2\,a_5\,a_8\,a_{13} - 8\,a_1\,a_6\,a_8\,a_{13} + 8\,a_0\,a_7\,a_8\,a_{13} - 4\,a_3^2\,a_9\,a_{13} -\\
&8\,a_2\,a_4\,a_9\,a_{13} + 24\,a_1\,a_5\,a_9\,a_{13} - 8\,a_0\,a_6\,a_9\,a_{13} + 8\,a_2\,a_3\,a_{10}\,a_{13} - 8\,a_1\,a_4\,a_{10}\,a_{13} -\\
&8\,a_0\,a_5\,a_{10}\,a_{13} + 4\,a_2^2\,a_{11}\,a_{13} - 8\,a_1\,a_3\,a_{11}\,a_{13} + 8\,a_0\,a_4\,a_{11}\,a_{13} - 4\,a_{10}^2\,a_{11}\,a_{13} + 4\,a_9\,a_{11}^2\,a_{13} -\\
&8\,a_1\,a_2\,a_{12}\,a_{13} + 8\,a_0\,a_3\,a_{12}\,a_{13} + 8\,a_9\,a_{10}\,a_{12}\,a_{13} - 8\,a_8\,a_{11}\,a_{12}\,a_{13} - 4\,a_7\,a_{12}^2\,a_{13} + 6\,a_1^2\,a_{13}^2 -\\
&4\,a_0\,a_2\,a_{13}^2 - 6\,a_9^2\,a_{13}^2 + 4\,a_8\,a_{10}\,a_{13}^2 + 4\,a_7\,a_{11}\,a_{13}^2 + 4\,a_6\,a_{12}\,a_{13}^2 - 4\,a_5\,a_{13}^3 - 4\,a_4\,a_5^2\,a_{14} +\\
&4\,a_4^2\,a_6\,a_{14} + 8\,a_3\,a_5\,a_6\,a_{14} - 12\,a_2\,a_6^2\,a_{14} - 8\,a_3\,a_4\,a_7\,a_{14} + 8\,a_2\,a_5\,a_7\,a_{14} + 8\,a_1\,a_6\,a_7\,a_{14} -\\
&4\,a_0\,a_7^2\,a_{14} - 4\,a_3^2\,a_8\,a_{14} + 8\,a_2\,a_4\,a_8\,a_{14} - 8\,a_1\,a_5\,a_8\,a_{14} + 8\,a_0\,a_6\,a_8\,a_{14} + 8\,a_2\,a_3\,a_9\,a_{14} -\\
&8\,a_1\,a_4\,a_9\,a_{14} - 8\,a_0\,a_5\,a_9\,a_{14} - 12\,a_2^2\,a_{10}\,a_{14} + 8\,a_1\,a_3\,a_{10}\,a_{14} + 8\,a_0\,a_4\,a_{10}\,a_{14} +\\
&4\,a_{10}^3\,a_{14} + 8\,a_1\,a_2\,a_{11}\,a_{14} - 8\,a_0\,a_3\,a_{11}\,a_{14} - 8\,a_9\,a_{10}\,a_{11}\,a_{14} + 4\,a_8\,a_{11}^2\,a_{14} - 4\,a_1^2\,a_{12}\,a_{14} +\\
&8\,a_0\,a_2\,a_{12}\,a_{14} + 4\,a_9^2\,a_{12}\,a_{14} - 8\,a_8\,a_{10}\,a_{12}\,a_{14} + 8\,a_7\,a_{11}\,a_{12}\,a_{14} - 4\,a_6\,a_{12}^2\,a_{14} -\\
&8\,a_0\,a_1\,a_{13}\,a_{14} + 8\,a_8\,a_9\,a_{13}\,a_{14} - 8\,a_7\,a_{10}\,a_{13}\,a_{14} - 8\,a_6\,a_{11}\,a_{13}\,a_{14} + 8\,a_5\,a_{12}\,a_{13}\,a_{14} +\\
&4\,a_4\,a_{13}^2\,a_{14} + 2\,a_0^2\,a_{14}^2 - 2\,a_8^2\,a_{14}^2 - 4\,a_7\,a_9\,a_{14}^2 + 12\,a_6\,a_{10}\,a_{14}^2 - 4\,a_5\,a_{11}\,a_{14}^2 - 4\,a_4\,a_{12}\,a_{14}^2 -\\
&4\,a_3\,a_{13}\,a_{14}^2 + 4\,a_2\,a_{14}^3 + 4\,a_4^2\,a_5\,a_{15} - 4\,a_3\,a_5^2\,a_{15} - 8\,a_3\,a_4\,a_6\,a_{15} + 8\,a_2\,a_5\,a_6\,a_{15} +\\
&4\,a_1\,a_6^2\,a_{15} + 12\,a_3^2\,a_7\,a_{15} - 8\,a_2\,a_4\,a_7\,a_{15} - 8\,a_1\,a_5\,a_7\,a_{15} - 8\,a_0\,a_6\,a_7\,a_{15} - 8\,a_2\,a_3\,a_8\,a_{15} +\\
&8\,a_1\,a_4\,a_8\,a_{15} + 8\,a_0\,a_5\,a_8\,a_{15} + 4\,a_2^2\,a_9\,a_{15} - 8\,a_1\,a_3\,a_9\,a_{15} + 8\,a_0\,a_4\,a_9\,a_{15} + 8\,a_1\,a_2\,a_{10}\,a_{15} -\\
&8\,a_0\,a_3\,a_{10}\,a_{15} - 4\,a_9\,a_{10}^2\,a_{15} - 4\,a_1^2\,a_{11}\,a_{15} - 8\,a_0\,a_2\,a_{11}\,a_{15} + 4\,a_9^2\,a_{11}\,a_{15} + 8\,a_8\,a_{10}\,a_{11}\,a_{15} -\\
&12\,a_7\,a_{11}^2\,a_{15} + 8\,a_0\,a_1\,a_{12}\,a_{15} - 8\,a_8\,a_9\,a_{12}\,a_{15} + 8\,a_7\,a_{10}\,a_{12}\,a_{15} + 8\,a_6\,a_{11}\,a_{12}\,a_{15} -\\
&4\,a_5\,a_{12}^2\,a_{15} + 4\,a_0^2\,a_{13}\,a_{15} - 4\,a_8^2\,a_{13}\,a_{15} + 8\,a_7\,a_9\,a_{13}\,a_{15} - 8\,a_6\,a_{10}\,a_{13}\,a_{15} + 8\,a_5\,a_{11}\,a_{13}\,a_{15} -\\
&8\,a_4\,a_{12}\,a_{13}\,a_{15} + 4\,a_3\,a_{13}^2\,a_{15} + 8\,a_7\,a_8\,a_{14}\,a_{15} - 8\,a_6\,a_9\,a_{14}\,a_{15} - 8\,a_5\,a_{10}\,a_{14}\,a_{15} +\\
&8\,a_4\,a_{11}\,a_{14}\,a_{15} + 8\,a_3\,a_{12}\,a_{14}\,a_{15} - 8\,a_2\,a_{13}\,a_{14}\,a_{15} - 4\,a_1\,a_{14}^2\,a_{15} - 6\,a_7^2\,a_{15}^2 + 4\,a_6\,a_8\,a_{15}^2 +\\
&4\,a_5\,a_9\,a_{15}^2 + 4\,a_4\,a_{10}\,a_{15}^2 - 12\,a_3\,a_{11}\,a_{15}^2 + 4\,a_2\,a_{12}\,a_{15}^2 + 4\,a_1\,a_{13}\,a_{15}^2 + 4\,a_0\,a_{14}\,a_{15}^2 + a_{15}^4\Big)
\end{aligned}
$$

Numbers of terms in the coefficients of this polynomial are :

```
Table[Length[Coefficient[prod1mod8, ξ, i]], {i, 0, 3}]
```

{245, 245, 245, 245}

```
prod3mod8 = Collect[
   Expand[MyProd[prod3mod16, prod11mod16] /. ruleωξ] /. ruleξ1 /. ruleξ2, ξ];
```

```
prod5mod8 = Collect[
   Expand[MyProd[prod5mod16, prod13mod16] /. ruleωξ] /. ruleξ1 /. ruleξ2, ξ];
```

```
prod7mod8 = Collect[
   Expand[MyProd[prod7mod16, prod15mod16] /. ruleωξ] /. ruleξ1 /. ruleξ2, ξ];
```

Each multiplication of two of these gives a linear polynomial of η :

```
prod1mod4 =
 Collect[Expand[MyProd[prod1mod8, prod5mod8] /. ruleξη] /. ruleη1 /. ruleη2, η]
```

$$
\begin{aligned}
&a_0^8 - a_2^8 + \langle\!\langle 45\,999 \rangle\!\rangle + 8\,a_7\,a_{15}^7 +\\
&\eta\,\Big(-a_1^8 + 8\,a_0\,a_1^6\,a_2 - 20\,a_0^2\,a_1^4\,a_2^2 + 16\,a_0^3\,a_1^2\,a_2^3 - 2\,a_0^4\,a_2^4 - 8\,a_0^2\,a_1^5\,a_3 + \langle\!\langle 45\,987 \rangle\!\rangle +\\
&\quad 8\,a_4\,a_{10}\,a_{15}^6 + 8\,a_3\,a_{11}\,a_{15}^6 + 8\,a_2\,a_{12}\,a_{15}^6 + 8\,a_1\,a_{13}\,a_{15}^6 + 8\,a_0\,a_{14}\,a_{15}^6 + a_{15}^8\Big)
\end{aligned}
$$

Numbers of terms in the coefficients of this polynomial are :

```
Table[Length[Coefficient[prod1mod4, η, i]], {i, 0, 1}]
```

{30 667, 30 667}

```
prod3mod4 =
  Collect[Expand[MyProd[prod3mod8, prod7mod8] /. ruleξη] /. ruleη1 /. ruleη2, η];
```

The maximum value of the coefficients of those terms is 896 as following:

```
tmp = Coefficient[prod3mod4, η, 0]
```

$a_0^8 - a_2^8 + 8\, a_1\, a_2^6\, a_3 - 20\, a_1^2\, a_2^4\, a_3^2 - 8\, a_0\, a_2^5\, a_3^2 + 16\, a_1^3\, a_2^2\, a_3^3 + \cdots 45\,982 \cdots + 8\, a_2\, a_4\, a_{15}^6 + 8\, a_1\, a_5\, a_{15}^6 + 8\, a_0\, a_6\, a_{15}^6 - 4\, a_{11}^2\, a_{15}^6 - 8\, a_{10}\, a_{12}\, a_{15}^6 - 8\, a_9\, a_{13}\, a_{15}^6 - 8\, a_8\, a_{14}\, a_{15}^6 + 8\, a_7\, a_{15}^7$

```
Abs[MonomialList[tmp] /. Table[a_i → 1, {i, 0, 15}]]
```

{1, 8, 8, 8, 8, 8, 8, 8, 28, 8, 16, 16, 16, 16, 48, 8, 8, 8, 16, 16, 16, 16, 48, 8, 16, 48, 16, 16, 24, 16, 8, 16, 8, 8, 16, 16, 16, 16, 8, 16, 16, 16, … 30 583 …, 32, 16, 48, 24, 16, 16, 32, 8, 8, 8, 16, 8, 4, 16, 8, 8, 24, 12, 24, 2, 8, 32, 16, 8, 20, 4, 8, 16, 8, 24, 8, 1, 8, 12, 24, 32, 8, 2, 16, 20, 8, 1}

```
Max[%]
```

896

Multiplication of these gives a non-negative real number:

```
norm4 = MyProd[prod1mod4, prod3mod4] /. ruleη2
```

$\left(a_0^8 - a_2^8 + \cdots 45\,999 \cdots + 8\, a_7\, a_{15}^7\right)^2 + \left(a_1^8 - 8\, a_0\, a_1^6\, a_2 + \cdots 45\,995 \cdots\right)^2$

Computation of me for e=2

Ne $(b_0(\kappa_0) - a(\kappa))$ with $\kappa \in S$

```
tbl21 = Flatten[Table[{a₀, a₁, a₂, a₃, norm2},
    {a₀, -1, 1, 2}, {a₁, 0, 2, 2}, {a₂, 0, 2, 2}, {a₃, 0, 2, 2}], 3];
```

```
TableForm[tbl21]
```

-1	0	0	0	1
-1	0	0	2	17
-1	0	2	0	25
-1	0	2	2	9
-1	2	0	0	17
-1	2	0	2	81
-1	2	2	0	73
-1	2	2	2	41
1	0	0	0	1
1	0	0	2	17
1	0	2	0	25
1	0	2	2	73
1	2	0	0	17
1	2	0	2	81
1	2	2	0	9
1	2	2	2	41

This part gives 16 norm values and their max is 81 as follows.

```
tbl21 = Flatten[Take[Transpose[tbl21], -1]]
```

{1, 17, 25, 9, 17, 81, 73, 41, 1, 17, 25, 73, 17, 81, 9, 41}

```
Length[tbl21]
```

16

```
Max[tbl21]
```

81

Ne $(b_0(\kappa_0) - b_i(\kappa))$ with $\kappa \in S$, $i = 0$

```
tbl22 = Drop[Flatten[Table[{a0, a1, a2, a3, norm2},
      {a0, 0, 0}, {a1, 0, 2, 2}, {a2, 0, 2, 2}, {a3, 0, 2, 2}], 3], 1];
```

(The case $\kappa = \kappa_0$ was deleted by the function Drop[..., 1].)

```
TableForm[tbl22]
```

0	0	0	2	16
0	0	2	0	16
0	0	2	2	32
0	2	0	0	16
0	2	0	2	64
0	2	2	0	32
0	2	2	2	16

This part gives 7 norm values and their max is 64 as follows.

```
tbl22 = Flatten[Take[Transpose[tbl22], -1]]
```

{16, 16, 32, 16, 64, 32, 16}

```
Length[tbl22]
```

7

```
Max[tbl22]
```

64

Ne $(b_0(\kappa_0) - b_i(\kappa))$ with $\kappa \in S$, $i \neq 0$

```
tbl23 = Flatten[
   Table[
    j = Mod[i, 3] + 1;
    k = Mod[j, 3] + 1;
    Table[{a0, a1, a2, a3, norm2},
      {a0, -1, 1, 2}, {ai, 1, 1, 0}, {aj, 0, 2, 2}, {ak, 0, 2, 2}],
    {i, 1, 3}]
   , 4];
```

```
TableForm[tbl23]
```

-1	1	0	0	2
-1	1	0	2	34
-1	1	2	0	34
-1	1	2	2	2
1	1	0	0	2
1	1	0	2	34
1	1	2	0	18
1	1	2	2	50
-1	0	1	0	4
-1	2	1	0	36
-1	0	1	2	4
-1	2	1	2	68
1	0	1	0	4
1	2	1	0	4
1	0	1	2	36
1	2	1	2	68
-1	0	0	1	2
-1	0	2	1	18
-1	2	0	1	34
-1	2	2	1	50
1	0	0	1	2
1	0	2	1	34
1	2	0	1	34
1	2	2	1	2

This part gives 24 norm values and their max is 68 as follows.

```
tbl23 = Flatten[Take[Transpose[tbl23], -1]]
```

```
{2, 34, 34, 2, 2, 34, 18, 50, 4, 36, 4, 68, 4, 4, 36, 68, 2, 18, 34, 50, 2, 34, 34, 2}
```

```
Length[tbl23]
```

```
24
```

```
Max[tbl23]
```

```
68
```

Summary: 47 norm values (17 different values). Their max is me=81

```
Length[tbl21] + Length[tbl22] + Length[tbl23]
```

```
47
```

```
Max[tbl21, tbl22, tbl23]
```

```
81
```

```
Union[tbl21, tbl22, tbl23]
```

```
{1, 2, 4, 9, 16, 17, 18, 25, 32, 34, 36, 41, 50, 64, 68, 73, 81}
```

```
Length[%]
```

```
17
```

Computation of me for e=3

Ne (b0 (κ0) - a (κ)) with κ ∈ S

```
tbl31 = Flatten[Table[{a0, a1, a2, a3, a4, a5, a6, a7, norm3},
    {a0, -1, 1, 2}, {a1, 0, 2, 2}, {a2, 0, 2, 2}, {a3, 0, 2, 2},
    {a4, 0, 2, 2}, {a5, 0, 2, 2}, {a6, 0, 2, 2}, {a7, 0, 2, 2}], 7];
```

This part gives 256 norm values and their max is 97393 as follows.

```
norms31 = Flatten[Take[Transpose[tbl31], -1]];
```

```
Length[norms31]
```

```
256
```

```
max31 = Max[norms31]
```

```
97 393
```

The maximizer patterns are :

```
Select[tbl31, #[[9]] == max31 &]
```

```
{{-1, 2, 2, 0, 2, 0, 2, 2, 97 393}, {1, 2, 2, 0, 2, 0, 2, 2, 97 393}}
```

Ne (b0 (κ0) - bi (κ)) with κ ∈ S, i = 0

```
tbl32 = Drop[Flatten[Table[{a0, a1, a2, a3, a4, a5, a6, a7, norm3},
     {a0, 0, 0}, {a1, 0, 2, 2}, {a2, 0, 2, 2}, {a3, 0, 2, 2},
     {a4, 0, 2, 2}, {a5, 0, 2, 2}, {a6, 0, 2, 2}, {a7, 0, 2, 2}], 7], 1];
```

(The case $\kappa = \kappa_0$ was deleted by the function Drop[..., 1].)

This part gives 127 norm values and their max is 73984 as follows.

```
norms32 = Flatten[Take[Transpose[tbl32], -1]];
```

```
Length[norms32]
```

```
127
```

```
Max[norms32]
```

```
73 984
```

Ne (b0 (κ0) - bi (κ)) with κ ∈ S, i ≠ 0

```
tbl33 = Flatten[
   Table[
    j1 = Mod[i, 7] + 1;
    j2 = Mod[j1, 7] + 1;
    j3 = Mod[j2, 7] + 1;
    j4 = Mod[j3, 7] + 1;
    j5 = Mod[j4, 7] + 1;
    j6 = Mod[j5, 7] + 1;
    Table[{a0, a1, a2, a3, a4, a5, a6, a7, norm3},
     {a0, -1, 1, 2}, {ai, 1, 1, 0}, {aj1, 0, 2, 2}, {aj2, 0, 2, 2},
     {aj3, 0, 2, 2}, {aj4, 0, 2, 2}, {aj5, 0, 2, 2}, {aj6, 0, 2, 2}],
    {i, 1, 7}]
   , 8];
```

This part gives 896 norm values and their max is 65314 as follows.

```
norms33 = Flatten[Take[Transpose[tbl33], -1]];
```

```
Length[norms33]
```

```
896
```

```
Max[norms33]
```

```
65 314
```

Summary: 1279 norm values (267 different values). Their max is me=97393

```
Length[norms31] + Length[norms32] + Length[norms33]
```

```
1279
```

```
Max[norms31, norms32, norms33]
```

```
97 393
```

```
Union[norms31, norms32, norms33]
```

```
{1, 2, 4, 16, 49, 81, 98, 113, 162, 196, 226, 256, 257, 272, 289, 324, 337, 353,
 452, 512, 514, 529, 577, 578, 593, 625, 674, 706, 784, 881, 1024, 1028, 1058,
 1156, 1186, 1201, 1217, 1250, 1296, 1348, 1412, 1552, 1649, 1681, 1762, 1777,
 1808, 1922, 2048, 2116, 2129, 2273, 2306, 2308, 2372, 2401, 2402, 2434, 2500,
 2657, 2833, 3088, 3106, 3202, 3217, 3281, 3298, 3313, 3361, 3362, 3524, 3554,
 3844, 3856, 3969, 4096, 4097, 4112, 4177, 4226, 4258, 4352, 4418, 4546, 4612,
 4624, 4802, 4804, 4868, 4996, 5314, 5329, 5378, 5392, 5537, 5569, 5648, 5666,
 6178, 6212, 6274, 6404, 6434, 6449, 6561, 6562, 6596, 6722, 6817, 6928, 7108,
 7121, 7184, 7556, 7633, 7793, 7938, 8002, 8081, 8098, 8192, 8194, 8273, 8452,
 8516, 8546, 8578, 8704, 8836, 9092, 9232, 9314, 9377, 9409, 9442, 9604, 9697,
 9986, 10 256, 10 466, 10 658, 10 756, 10 768, 10 786, 10 897, 11 074, 11 441,
 11 969, 12 241, 12 482, 12 544, 12 769, 13 058, 13 073, 13 122, 13 124, 13 328,
 13 378, 13 444, 13 634, 14 096, 14 161, 14 242, 14 321, 14 722, 14 737, 14 786,
 14 864, 15 044, 15 073, 15 121, 15 266, 15 632, 15 793, 16 004, 16 144, 16 162,
 16 384, 16 388, 16 609, 17 092, 17 153, 17 408, 17 924, 18 052, 18 306, 18 562,
 18 721, 18 818, 18 884, 19 202, 19 204, 19 472, 19 984, 20 674, 20 736, 20 738,
 21 188, 21 776, 21 794, 22 032, 22 049, 22 148, 22 658, 22 721, 22 882, 23 409,
 23 633, 23 938, 23 953, 24 832, 25 088, 25 313, 25 538, 25 889, 25 921, 26 146,
 26 384, 27 073, 27 268, 28 273, 28 322, 28 817, 28 928, 29 057, 29 186, 29 444,
 29 794, 30 434, 30 532, 32 272, 32 644, 32 657, 32 866, 33 026, 33 218, 34 289,
 34 306, 34 816, 35 153, 36 368, 36 612, 37 409, 40 177, 40 708, 40 738, 42 001,
 43 202, 44 098, 46 754, 47 617, 49 664, 51 697, 51 938, 52 496, 54 401, 54 532,
 55 681, 56 002, 59 716, 60 688, 63 236, 65 201, 65 314, 73 984, 77 537, 97 393}
```

```
Length[%]
```

```
267
```

Computation of me for e=4

Ne (b_0 (κ_0) − a (κ)) with $\kappa \in$ S

```
tbl41 = Flatten[
   Table[{a0, a1, a2, a3, a4, a5, a6, a7, a8, a9, a10, a11, a12, a13, a14, a15, norm4},
    {a0, -1, 1, 2}, {a1, 0, 2, 2}, {a2, 0, 2, 2}, {a3, 0, 2, 2},
    {a4, 0, 2, 2}, {a5, 0, 2, 2}, {a6, 0, 2, 2}, {a7, 0, 2, 2},
    {a8, 0, 2, 2}, {a9, 0, 2, 2}, {a10, 0, 2, 2}, {a11, 0, 2, 2},
    {a12, 0, 2, 2}, {a13, 0, 2, 2}, {a14, 0, 2, 2}, {a15, 0, 2, 2}], 15];
```

This part gives 65536 norm values and their max is 542909003297 as follows.

```
norms41 = Flatten[Take[Transpose[tbl41], -1]];
```

```
Length[norms41]
```

```
65 536
```

```
max41 = Max[norms41]
```

```
542 909 003 297
```

The maximizer patterns are :

```
Select[tbl41, #[[17]] == max41 &]
```

```
{{-1, 2, 2, 2, 2, 0, 0, 2, 2, 0, 0, 2, 0, 2, 0, 2, 542 909 003 297},
 {1, 2, 0, 2, 0, 2, 0, 0, 2, 2, 0, 0, 2, 2, 2, 2, 542 909 003 297}}
```

Thus, the max of the norm in e = 4 is expected to be 542 909 003 297.