Appendix: Reduction of computation times of $N_e(\cdot)$s

(i) For $\alpha = a(\kappa) - b_0(\kappa_0)$,

$$\alpha = (\pm 1, \pm 1, \cdots, \pm 1) - (0, 1, \cdots, 1) = (\pm 1, \alpha_1, \cdots, \alpha_{2^e-1}),$$

where $\alpha_i \in \{0, -2\}$. Number of α is 2^{2^e}.

(ii) For $\alpha = b_i(\kappa) - b_0(\kappa_0)$ with $i = 0$ and $\kappa \neq \kappa_0$,

$$\alpha = (0, \pm 1, \cdots, \pm 1) - (0, 1, \cdots, 1) = (0, \alpha_1, \cdots, \alpha_{2^e-1}),$$

where $\alpha_i \in \{0, -2\}$ and at least one of α_i is not zero. Number of α is $2^{2^e-1} - 1$.

(iii) For $\alpha = b_i(\kappa) - b_0(\kappa_0)$ with $i \neq 0$,

$$\alpha = (\pm 1, \cdots \pm 1, 0, \pm 1, \cdots, \pm 1) - (0, 1, \cdots, 1) = (\pm 1, \alpha_1, \cdots, \alpha_{i-1}, -1, \alpha_{i+1}, \alpha_{2^e-1}),$$

where $\alpha_i \in \{0, -2\}$. Number of α is $(2^e - 1)(2^{2^e-1})$.

Total number of $N_e(\alpha)$ is

$$2^{2^e} + 2^{2^e-1} - 1 + (2^e - 1)(2^{2^e-1}) = 2^{2^e}(\frac{1}{2}2^e + 1) - 1.$$

Lemma 1 *Let n be an odd number.*

$$f_e(c, a_1, \cdots, a_n) = f_e(-c, a_n, \cdots, a_1)$$

proof: By transposing and multiplying $(-1)^{n+1}$,

$$(\text{LHS}) = \begin{vmatrix} c & a_1 & \cdots & a_n \\ -a_n & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_1 \\ -a_1 & \cdots & -a_n & c \end{vmatrix} = \begin{vmatrix} c & -a_n & \cdots & -a_1 \\ a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_n \\ a_n & \cdots & a_1 & c \end{vmatrix} = \begin{vmatrix} -c & a_n & \cdots & a_1 \\ -a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & a_n \\ -a_n & \cdots & -a_1 & -c \end{vmatrix} = (\text{RHS})$$

Corollary 1

$$N_e\Big(\{a(\kappa) - b_0(\kappa_0) | \kappa(0) = 1\}\Big) = N_e\Big(\{a(\kappa) - b_0(\kappa_0) | \kappa(0) = -1\}\Big)$$

proof: We use Lamma 1 as $c = 1$. For any $\alpha \in \{a(\kappa) - b_0(\kappa_0) | \kappa(0) = 1\}$,

$$N_e(\alpha) = f_e(1, a_1, \cdots, a_{2^e-1}) = f_e(-1, a_{2^e-1}, \cdots, a_1) = N_e(\alpha'),$$

where $a_i \in \{0, -2\}$, $i \in I_0$. This α' belongs to $\{a(\kappa) - b_0(\kappa_0) | \kappa(0) = -1\}$.

Corollary 2 *For any $i \in I_0$,*

$$N_e\Big(\{b_i(\kappa) - b_0(\kappa_0) | \kappa(0) = 1\}\Big) = N_e\Big(\{b_{2^e-i}(\kappa) - b_0(\kappa_0) | \kappa(0) = -1\}\Big)$$

proof: We use Lamma 1 as $c = 1$. For any $\alpha \in \{b_i(\kappa) - b_0(\kappa_0) | \kappa(0) = 1\}$,

$$\begin{aligned} N_e(\alpha) &= f_e(1, \quad a_1, \cdots, a_{i-1}, 1, a_{i+1}, \cdots, a_{2^e-1}) \\ &= f_e(-1, \quad a_{2^e-1}, \cdots, a_{i+1}, -1, a_{i-1}, \cdots, a_1) \\ &= N_e(\alpha'), \end{aligned}$$

where $a_i \in \{0, -2\}$, $i \in I_0 \cap I_i$. This α' belongs to $\{b_{2^e-i}(\kappa) - b_0(\kappa_0) | \kappa(0) = -1\}$.

Remark 1 *We can omit computations of $N_e(\alpha)$ when $\alpha_0 = 1$, which are half cases of (i) and (iii).*

Total number of α to be computed is reduced to

$$\frac{1}{2}2^{2^e} + 2^{2^e-1} - 1 + \frac{1}{2}(2^e - 1)(2^{2^e-1}) = 2^{2^e}(\frac{1}{4}2^e + \frac{3}{4}) - 1.$$

Corollary 3 *Let n be an odd number.*

$$f_e(0, a_1, \cdots, a_n) = f_e(0, a_n, \cdots, a_1)$$

Remark 2 *Wen can omit one computation of $f_e(0, a_1, \cdots, a_n)$ and $f_e(0, a_n, \cdots, a_1)$, which appear in (ii).*

Lemma 2 *Let n be an odd number.*

$$f_e(a_0, \cdots, a_n) = f_e(a_n, \cdots, a_0)$$

proof:

$$(\mathrm{LHS}) \overset{\text{Lemma 1}}{=} f_e(-a_0, a_n, \cdots, a_1) = \begin{vmatrix} -a_0 & a_n & \cdots & a_2 & a_1 \\ -a_1 & -a_0 & a_n & \cdots & a_2 \\ -a_2 & -a_1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & -a_0 & a_n \\ -a_n & \cdots & -a_2 & -a_1 & -a_0 \end{vmatrix}$$

$$= - \begin{vmatrix} a_n & \cdots & a_2 & a_1 & -a_0 \\ -a_0 & a_n & \cdots & a_2 & -a_1 \\ -a_1 & \ddots & \ddots & \vdots & -a_2 \\ \vdots & \ddots & -a_0 & a_n & \vdots \\ -a_{n-1} & \cdots & -a_1 & -a_0 & -a_n \end{vmatrix}$$

$$= \begin{vmatrix} a_n & \cdots & a_2 & a_1 & a_0 \\ -a_0 & a_n & \cdots & a_2 & a_1 \\ -a_1 & \ddots & \ddots & \vdots & a_2 \\ \vdots & \ddots & -a_0 & a_n & \vdots \\ -a_{n-1} & \cdots & -a_1 & -a_0 & a_n \end{vmatrix} = (\mathrm{RHS})$$

Remark 3 *For $\alpha \in \{b_{2^e-\mathbf{1}}(\kappa) - b_0(\kappa_0) | \kappa(0) = -1\}$ in (iii), $a_0 = a_{2^e-1}(= -1)$ holds. We therefore can omit one computation of $f_e(-1, a_1, \cdots, a_{2^e-2}, -1)$ and $f_e(-1, a_{2^e-2}, \cdots, a_1, -1)$.*

Corollary 4 *Let n be an odd number.*
(i) $f_e(a_1, \cdots, a_n, 0) = f_e(0, a_1, \cdots, a_n)$
(ii) $f_e(a_1, \cdots, a_n, 0) = f_e(a_n, \cdots, a_1, 0)$

proof:

$$\begin{array}{ccc} f_e(a_1, \cdots, a_n, 0) & \overset{\text{Lemma 2}}{=} & f_e(0, a_n, \cdots, a_1) \\ & & \| \;\text{Cor. 3} \\ f_e(a_n, \cdots, a_1, 0) & \overset{\text{Lemma 2}}{=} & f_e(0, a_1, \cdots, a_n) \end{array}$$

Corollary 5 *We can omit all computations of $N(\alpha)$ if $a_{2^e-1} = 0$ in (ii), that is, we omit $f_e(0, a_1, \cdots, a_{2^e-2}, 0)$.*

proof. In (ii), for any $\alpha = (0, a_1, \cdots, a_{2^e-1})$ in which $a_{2^e-1} = 0$, there exists one index j such that $a_j \neq 0$ and $a_{j+1} = \cdots = a_{2^e-1} = 0$. Then

$$f_e(0, a_1, \cdots, a_j, 0, \cdots, 0, 0) = f_e(0, 0, a_1, \cdots, a_j, 0, \cdots, 0) = \cdots = f_e(0, \cdots, 0, 0, 0, a_1, \cdots, a_j)$$

by virtue of Corollary 4.

In implementation, reduction by Remark 2, Remark 3 and Corollary 5 needs no-simple program codes. So these reduction is not applied for $e = 1, 2$ (only Remark 1 is applied there.)