

Number of α to compute the norm $\text{Ne}(\alpha)$ and its reductions

(i) For $\alpha = a(\kappa) - b_0(\kappa_0)$,

$$\alpha = (\pm 1, \pm 1, \dots, \pm 1) - (0, 1, \dots, 1) = (\pm 1, \alpha_1, \dots, \alpha_{2^e-1}),$$

where $\alpha_i \in \{0, -2\}$. Number of α is 2^{2^e} .

α_0 can be fixed to +1 by Remark1 of Appendix so that the number is reduced to $\frac{1}{2} 2^{2^e}$

(ii) For $\alpha = b_i(\kappa) - b_0(\kappa_0)$ with $i = 0$ and $\kappa \neq \kappa_0$,

$$\alpha = (0, \pm 1, \dots, \pm 1) - (0, 1, \dots, 1) = (0, \alpha_1, \dots, \alpha_{2^e-1}),$$

where $\alpha_i \in \{0, -2\}$ and at least one of α_i is not zero. Number of α is $2^{2^e-1} - 1$.

By Appendix the number is reduced as follows. Now, we rewrite $\{0, -2\}$ to $\{0, 1\}$.

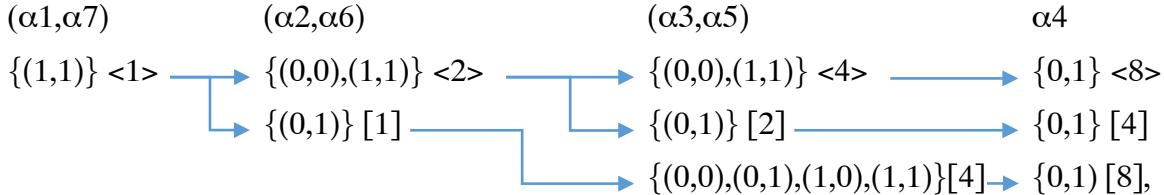
Rule-a: α_{2^e-1} can be fixed to 1.

Rule-b: $\text{Ne}(0, \dots, 0, \alpha_k, \dots, \alpha_{2^{e-1}}) = \text{Ne}(0, \dots, 0, \alpha_{2^{e-1}}, \dots, \alpha_k)$. Hence, we choose $\alpha = (0, \dots, 0, \alpha_k, \dots, \alpha_{2^{e-1}})$ if $(\alpha_k \dots \alpha_{2^{e-1}})_2 \leq (\alpha_{2^{e-1}} \dots \alpha_k)_2$ as binary numbers.

--- Number of sets of the arguments $\alpha_1, \dots, \alpha_{2^{e-1}}$ to be considered are analyzed by the following trees.

$e=3$

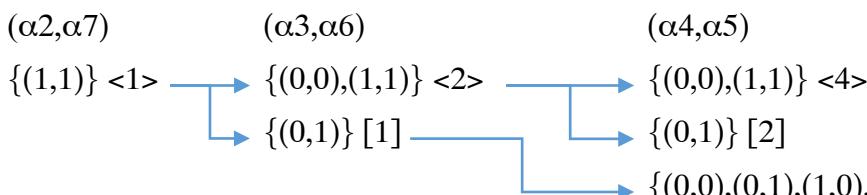
The tree when $\alpha_1=1$ is:



and it has totally 20 sets,

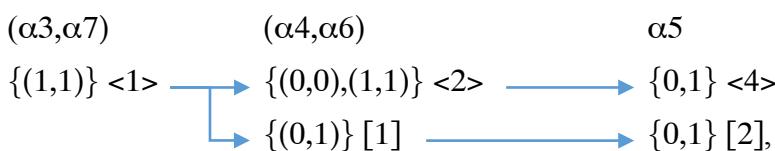
where numbers in parentheses $< >$ or $[]$ show the number of sets of the arguments. The tree node with the parenthesis $< >$ has possibility of $(\alpha_k \dots \alpha_{2^{e-1}})_2 = (\alpha_{2^{e-1}} \dots \alpha_k)_2$, so that Rule-b is also applied to its child nodes. The tree node with parenthesis $[]$ has lost the possibility, so that all values appear in the arguments on its child nodes.

The tree when $\alpha_1=0$ and $\alpha_2=1$ is:



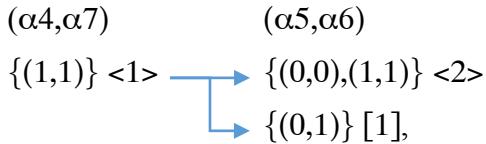
and it has totally 10 sets.

The tree when $\alpha_1=\alpha_2=0$ and $\alpha_3=1$ is:



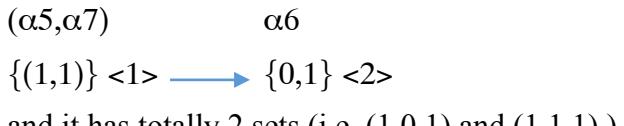
and it has totally 6 sets (i.e. (1,0,0,0,1), (1,0,1,0,1), (1,1,0,1,1), (1,1,1,1,1), (1,0,0,1,1) and (1,0,1,1,1)).

The tree when $\alpha_1=\alpha_2=\alpha_3=0$ and $\alpha_4=1$ is:



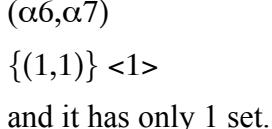
and it has totally 3 sets (i.e. $(1,0,0,1)$, $(1,1,1,1)$ and $(1,0,1,1)$).

The tree when $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ and $\alpha_5=1$ is:



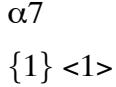
and it has totally 2 sets (i.e. $(1,0,1)$ and $(1,1,1)$).

The tree when $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=0$ and $\alpha_6=1$ is:



and it has only 1 set.

The tree when $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=0$ and $\alpha_7=1$ is:



and it has only 1 set.

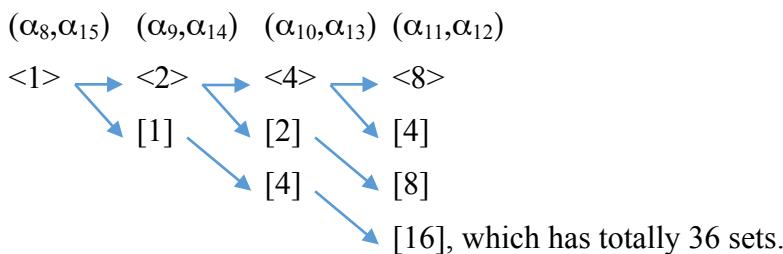
Total number of α to be considered when $e=3$ is $1+1+2+3+6+10+20=43$.

$e=4$

Now, $\alpha = (0, \alpha_1, \dots, \alpha_{15})$.

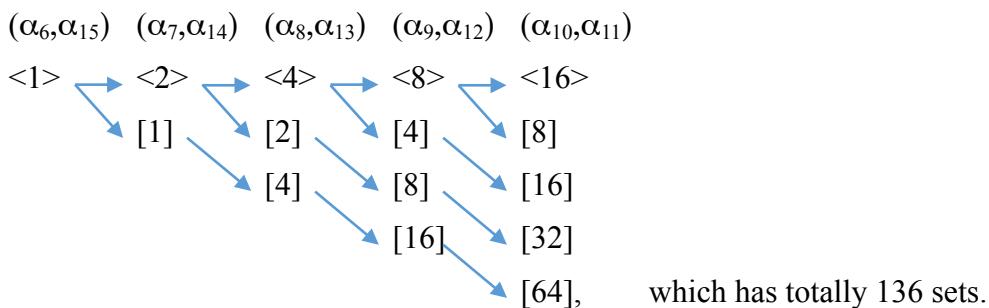
The trees when $\alpha_1=\dots=\alpha_8=0$ have 43 sets as same discussion as above.

The tree when $\alpha_1=\dots=\alpha_7=0$ and $\alpha_8=1$ is



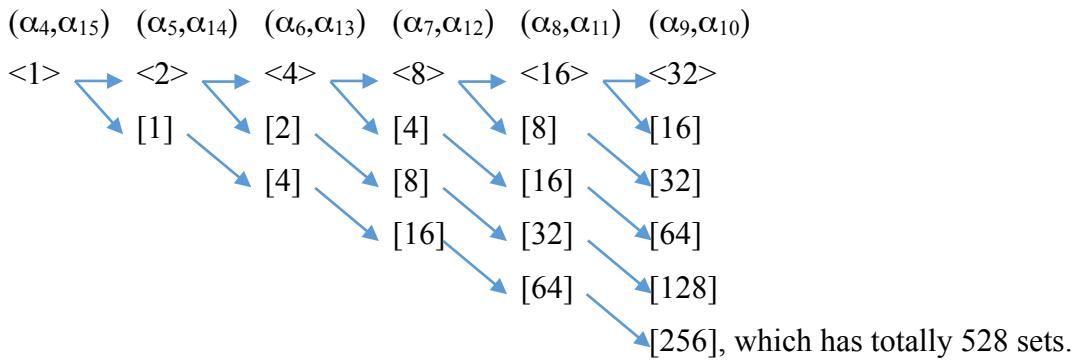
The tree when $\alpha_1=\dots=\alpha_6=0$ and $\alpha_7=1$ has twice of above, i.e. 72 sets.

The tree when $\alpha_1=\dots=\alpha_5=0$ and $\alpha_6=1$ is



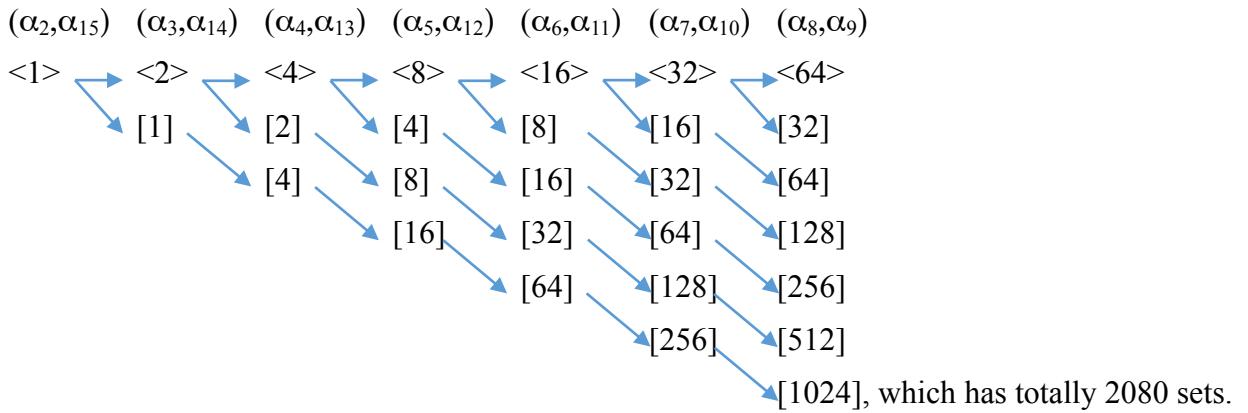
The tree when $\alpha_1=\dots=\alpha_4=0$ and $\alpha_5=1$ has twice of above, i.e. 272 sets.

The tree when $\alpha_1=\dots=\alpha_3=0$ and $\alpha_4=1$ is



The tree when $\alpha_1=\alpha_2=0$ and $\alpha_3=1$ has twice of above, i.e. 1056 sets.

The tree when $\alpha_1=0$ and $\alpha_2=1$ is



The tree when $\alpha_1=1$ has twice of above, i.e. 4160 sets.

Total number of α to be considered when $e=4$ is $43+36+72+136+272+528+1056+2080+4160=8383$.

For general e

"non-zero width"	$e=1$	$e=2$	$e=3$	$e=4$	$e \geq 5$
1	1	1	1	1	1
2	-	1	1	1	1
3	-	*	*	*	*
4	-	-	3	3	$3=2^1+2^0$
5	-	-	*	*	*
6	-	-	10	10	$10=2^3+2^1$
7	-	-	*	*	*
8	-	-	-	36	$36=2^5+2^2$
9	-	-	-	*	*
10	-	-	-	136	$136=2^7+2^3$
11	-	-	-	*	*
12	-	-	-	528	$528=2^9+2^4$
13	-	-	-	*	*
14	-	-	-	2080	$2080=2^{11}+2^5$
15	-	-	-	*	*
16	-	-	-	-	...

where * is the twice of the above.

Number of the sets for "non-zero width" ≥ 4 is

$$\begin{aligned} 3 \times \sum_{i=1}^{(2^{e-1}-2)} (2^{2i-1} + 2^{i-1}) &= 3 \times \left\{ 2 \cdot \frac{4^{(2^{e-1}-2)-1}}{4-1} + 1 \cdot \frac{2^{(2^{e-1}-2)-1}}{2-1} \right\} \\ &= 2 \cdot 4^{(2^{e-1}-2)} + 3 \cdot 2^{(2^{e-1}-2)} - 5. \end{aligned}$$

Total number of the sets is

$$\begin{aligned} 2 \cdot 4^{(2^{e-1}-2)} + 3 \cdot 2^{(2^{e-1}-2)} - 1 &= 2 \cdot 2^{(2^e-4)} + 3 \cdot 2^{(2^{e-1}-2)} - 1 \\ &= 2^{(2^e-3)} + 3 \cdot 2^{(2^{e-1}-2)} - 1 \end{aligned}$$

(iii) For $\alpha = b_i(\kappa) - b_0(\kappa_0)$ with $i \neq 0$,

$$\alpha = (\pm 1, \dots, \pm 1, 0, \pm 1, \dots, \pm 1) - (0, 1, \dots, 1) = (\pm 1, \alpha_1, \dots, \alpha_{i-1}, -1, \alpha_{i+1}, \alpha_{2^e-1}),$$

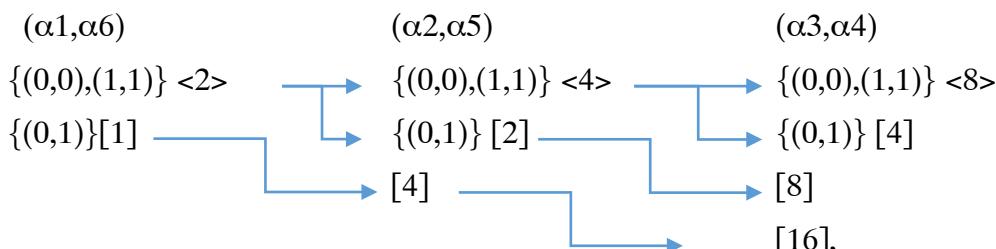
where $\alpha_i \in \{0, -2\}$. Number of α is $(2^e - 1)(2^{2^e-1})$.

By Remark1 of Appendix, α_0 can be fixed to -1 so that the number is reduced to $\frac{1}{2}(2^e - 1)2^{2^e-1}$

By Appendix the number is reduced further as follows.

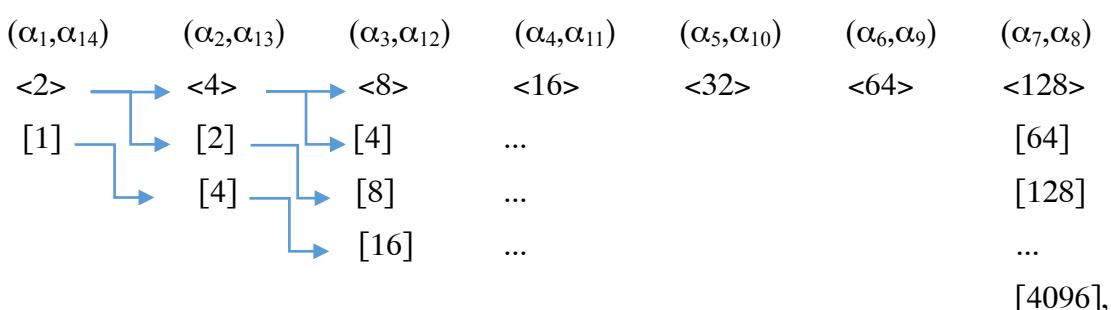
If $\alpha_{2^e-1} = -1$ then $\text{Ne}(-1, \alpha_1, \dots, \alpha_{2^e-2}, -1) = \text{Ne}(-1, \alpha_{2^e-2}, \dots, \alpha_1, -1)$. Now, we again rewrite $\{0, -2\}$ to $\{0, 1\}$. We choose $\alpha = (-1, \alpha_1, \dots, \alpha_{2^e-2}, -1)$ if $(\alpha_1, \dots, \alpha_{2^e-2})_2 \leq (\alpha_{2^e-2}, \dots, \alpha_1)_2$ as binary numbers. Number of sets of the arguments $\alpha_1, \dots, \alpha_{2^e-2}$ to be considered are analyzed by the following trees.

$e=3$



and it has totally 36 sets.

$e=4$



and it has totally 8256 sets.

For general e , number of the sets is $2^{2^e-3} + 2^{2^{e-1}-2}$.

Total number of sets in (iii) is obtained by adding $\frac{1}{2}(2^e - 2)2^{2^e-1}$ in the case of $\alpha_{2^e-1} \neq -1$, and the number is $2^{2^e+e-2} - 3 \cdot 2^{2^e-3} + 2^{2^{e-1}-2}$.

The total numbers of (i),(ii),(iii) are

(original)

$$2^{2^e} + (2^{2^{e-1}} - 1) + (2^e - 1)2^{2^{e-1}} = 2^{2^e+e-1} + 2^{2^e} - 1$$

(reduction by Remark 1)

$$\frac{1}{2}2^{2^e} + (2^{2^{e-1}} - 1) + \frac{1}{2}(2^e - 1)2^{2^{e-1}} = 2^{2^e+e-2} + \frac{3}{4}2^{2^e} - 1$$

(reduction by Appendix all)

$$\begin{aligned} \frac{1}{2}2^{2^e} + [2^{(2^{e-3})} + 3 \cdot 2^{(2^{e-1}-2)} - 1] + [2^{2^e+e-2} - 3 \cdot 2^{2^{e-3}} + 2^{2^{e-1}-2}] \\ = 2^{2^e+e-2} + 2^{2^{e-2}} + 2^{2^{e-1}} - 1 \end{aligned}$$