Number of α to compute the norm Ne(α) and its reductions

(i) For $\alpha = a(\kappa) - b_0(\kappa_0)$,

$$\alpha = (\pm1, \pm1, \cdots, \pm1) - (0, 1, \cdots, 1) = (\pm1, \alpha_1, \cdots, \alpha_{2^e-1}),$$

where $\alpha_i \in \{0, -2\}$. Number of α is 2^{2^e}.

α_0 can be fixed to +1 by Remark1 of Appendix so that the number is reduced to $\frac{1}{2}2^{2^e}$

(ii) For $\alpha = b_i(\kappa) - b_0(\kappa_0)$ with $i = 0$ and $\kappa \neq \kappa_0$,

$$\alpha = (0, \pm1, \cdots, \pm1) - (0, 1, \cdots, 1) = (0, \alpha_1, \cdots, \alpha_{2^e-1}),$$

where $\alpha_i \in \{0, -2\}$ and at least one of α_i is not zero. Number of α is $2^{2^e-1} - 1$.

By Appendix the number is reduced as follows. Now, we rewrite {0, -2} to {0, 1}.

Rule-a: $\alpha_{2^e\text{-}1}$ can be fixed to 1.

Rule-b: Ne(0, ..., 0, α_k, ..., $\alpha_{2\text{^}e\text{-}1}$) = Ne(0, ..., 0, $\alpha_{2\text{^}e\text{-}1}$, ..., α_k). Hence, we choose α = (0, ..., 0, α_k, ..., $\alpha_{2\text{^}e\text{-}1}$) if $(\alpha_k \ldots \alpha_{2\text{^}e\text{-}1})_2$ <= $(\alpha_{2\text{^}e\text{-}1} \ldots \alpha_k)_2$ as binary numbers.

--- Number of sets of the arguments α_1, ..., $\alpha_{2\text{^}e\text{-}1}$ to be considered are analyzed by the following trees.

e=3

The tree when α_1=1 is:

(α1,α7)	(α2,α6)	(α3,α5)	α4
{(1,1)} <1> →	{(0,0),(1,1)} <2> →	{(0,0),(1,1)} <4> →	{0,1} <8>
	{(0,1)} [1] →	{(0,1)} [2] →	{0,1} [4]
		{(0,0),(0,1),(1,0),(1,1)}[4] →	{0,1) [8],

and it has totally 20 sets,

where numbers in parentheses < > or [ ] show the number of sets of the arguments. The tree node with the parenthesis < > has possibility of $(\alpha_k \ldots \alpha_{2\text{^}e\text{-}1})_2 = (\alpha_{2\text{^}e\text{-}1} \ldots \alpha_k)_2$, so that Rule-b is also applied to its child nodes. The tree node with parenthesis [ ] has lost the possibility, so that all values appear in the arguments on its child nodes.

The tree when α_1=0 and α_2=1 is:

(α2,α7)	(α3,α6)	(α4,α5)
{(1,1)} <1> →	{(0,0),(1,1)} <2> →	{(0,0),(1,1)} <4>
	{(0,1)} [1] →	{(0,1)} [2]
		{(0,0),(0,1),(1,0),(1,1)}[4], and it has totally 10 sets.

The tree when α_1=α_2=0 and α_3=1 is:

(α3,α7)	(α4,α6)	α5
{(1,1)} <1> →	{(0,0),(1,1)} <2> →	{0,1} <4>
	{(0,1)} [1] →	{0,1} [2],

and it has totally 6 sets (i.e. (1,0,0,0,1), (1,0,1,0,1), (1,1,0,1,1), (1,1,1,1,1), (1,0,0,1,1) and (1,0,1,1,1)).

The tree when $\alpha_1=\alpha_2=\alpha_3=0$ and $\alpha_4=1$ is:

(α4,α7) (α5,α6)

{(1,1)} <1> → {(0,0),(1,1)} <2>

→ {(0,1)} [1],

and it has totally 3 sets (i.e. (1,0,0,1), (1,1,1,1) and (1,0,1,1)).

The tree when $\alpha_1=\alpha_2=\alpha_3=\alpha_4=0$ and $\alpha_5=1$ is:

(α5,α7) α6

{(1,1)} <1> → {0,1} <2>

and it has totally 2 sets (i.e. (1,0,1) and (1,1,1)).

The tree when $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=0$ and $\alpha_6=1$ is:

(α6,α7)

{(1,1)} <1>

and it has only 1 set.

The tree when $\alpha_1=\alpha_2=\alpha_3=\alpha_4=\alpha_5=\alpha_6=0$ and $\alpha_7=1$ is:

α7

{1} <1>

and it has only 1 set.

Total number of α to be considered when e=3 is 1+1+2+3+6+10+20=43.

e=4

Now, $\alpha = (0, \alpha_1, ..., \alpha_{15})$.

The trees when $\alpha_1=...=\alpha_8=0$ have 43 sets as same discussion as above.

The tree when $\alpha_1=...=\alpha_7=0$ and $\alpha_8=1$ is

(α_8,α_{15}) (α_9,α_{14}) (α_{10},α_{13}) (α_{11},α_{12})

<1> → <2> → <4> → <8>

[1] [2] [4]

[4] [8]

[16], which has totally 36 sets.

The tree when $\alpha_1=...=\alpha_6=0$ and $\alpha_7=1$ has twice of above, i.e. 72 sets.

The tree when $\alpha_1=...=\alpha_5=0$ and $\alpha_6=1$ is

(α_6,α_{15}) (α_7,α_{14}) (α_8,α_{13}) (α_9,α_{12}) (α_{10},α_{11})

<1> → <2> → <4> → <8> → <16>

[1] [2] [4] [8]

[4] [8] [16]

[16] [32]

[64], which has totally 136 sets.

The tree when $\alpha_1=...=\alpha_4=0$ and $\alpha_5=1$ has twice of above, i.e. 272 sets.

The tree when $\alpha_1=...=\alpha_3=0$ and $\alpha_4=1$ is

(α_4,α_{15})	(α_5,α_{14})	(α_6,α_{13})	(α_7,α_{12})	(α_8,α_{11})	(α_9,α_{10})
<1>	<2>	<4>	<8>	<16>	<32>
	[1]	[2]	[4]	[8]	[16]
		[4]	[8]	[16]	[32]
			[16]	[32]	[64]
				[64]	[128]
					[256], which has totally 528 sets.

The tree when $\alpha_1=\alpha_2=0$ and $\alpha_3=1$ has twice of above, i.e. 1056 sets.

The tree when $\alpha_1=0$ and $\alpha_2=1$ is

(α_2,α_{15})	(α_3,α_{14})	(α_4,α_{13})	(α_5,α_{12})	(α_6,α_{11})	(α_7,α_{10})	(α_8,α_9)
<1>	<2>	<4>	<8>	<16>	<32>	<64>
	[1]	[2]	[4]	[8]	[16]	[32]
		[4]	[8]	[16]	[32]	[64]
			[16]	[32]	[64]	[128]
				[64]	[128]	[256]
					[256]	[512]
						[1024], which has totally 2080 sets.

The tree when $\alpha_1=1$ has twice of above, i.e. 4160 sets.

Total number of α to be considered when e=4 is 43+36+72+136+272+528+1056+2080+4160=8383.

For general e

"non-zero width"	e=1	e=2	e=3	e=4	e>=5
1	1	1	1	1	1
2	-	1	1	1	1
3	-	*	*	*	*
4	-	-	3	3	$3=2^1+2^0$
5	-	-	*	*	*
6	-	-	10	10	$10=2^3+2^1$
7	-	-	*	*	*
8	-	-	-	36	$36=2^5+2^2$
9	-	-	-	*	*
10	-	-	-	136	$136=2^7+2^3$
11	-	-	-	*	*
12	-	-	-	528	$528=2^9+2^4$
13	-	-	-	*	*
14	-	-	-	2080	$2080=2^{11}+2^5$
15	-	-	-	*	*
16	-	-	-	-	...

where * is the twice of the above.

Number of the sets for "non-zero width" >= 4 is

$$3\times \sum_{i=1}^{(2^{e-1}-2)} (2^{2i-1}+2^{i-1}) = 3\times\left\{2\cdot\frac{4^{(2^{e-1}-2)}-1}{4-1} + 1\cdot\frac{2^{(2^{e-1}-2)}-1}{2-1}\right\}$$

$$= 2\cdot 4^{(2^{e-1}-2)} + 3\cdot 2^{(2^{e-1}-2)} - 5.$$

Total number of the sets is

$$2\cdot 4^{(2^{e-1}-2)} + 3\cdot 2^{(2^{e-1}-2)} - 1 = 2\cdot 2^{(2^{e}-4)} + 3\cdot 2^{(2^{e-1}-2)} - 1$$

$$= 2^{(2^{e}-3)} + 3\cdot 2^{(2^{e-1}-2)} - 1$$

(iii) For $\alpha = b_i(\kappa) - b_0(\kappa_0)$ with $i \neq 0$,

$$\alpha = (\pm 1, \cdots \pm 1, 0, \pm 1, \cdots, \pm 1) - (0, 1, \cdots, 1) = (\pm 1, \alpha_1, \cdots, \alpha_{i-1}, -1, \alpha_{i+1}, \alpha_{2^e-1}),$$

where $\alpha_i \in \{0, -2\}$. Number of α is $(2^e - 1)(2^{2^e-1})$.

By Remark1 of Appendix, α_0 can be fixed to -1 so that the number is reduced to $\frac{1}{2}(2^e-1)2^{2^e-1}$

By Appendix the number is reduced further as follows.

If $\alpha_{2^e\text{-}1} = -1$ then Ne(-1, α_1, ..., $\alpha_{2\text{^}e\text{-}2}$, -1) = Ne(-1, $\alpha_{2\text{^}e\text{-}2}$, ..., α_1, -1). Now, we again rewrite {0, -2} to {0, 1}. We choose α = (-1, α_1, ..., $\alpha_{2\text{^}e\text{-}2}$, -1) if $(\alpha_1, ..., \alpha_{2\text{^}e\text{-}2})_2$ <= $(\alpha_{2\text{^}e\text{-}2}, ..., \alpha_1)_2$ as binary numbers. Number of sets of the arguments α_1, ..., $\alpha_{2\text{^}e\text{-}2}$ to be considered are analyzed by the following trees.

e=3

(α1,α6)	(α2,α5)	(α3,α4)
{(0,0),(1,1)} <2>	{(0,0),(1,1)} <4>	{(0,0),(1,1)} <8>
{(0,1)}[1]	{(0,1)} [2]	{(0,1)} [4]
	[4]	[8]
		[16],

and it has totally 36 sets.

e=4

(α_1,α_{14})	(α_2,α_{13})	(α_3,α_{12})	(α_4,α_{11})	(α_5,α_{10})	(α_6,α_9)	(α_7,α_8)
<2>	<4>	<8>	<16>	<32>	<64>	<128>
[1]	[2]	[4]	...			[64]
	[4]	[8]	...			[128]
		[16]	...			...
						[4096],

and it has totally 8256 sets.

For general e, number of the sets is $2^{2^e-3} + 2^{2^{e-1}-2}$.

Total number of sets in (iii) is obtained by adding $\frac{1}{2}(2^e-2)2^{2^e-1}$ in the case of $\alpha_{2^e\text{-}1} \neq -1$, and the number is $2^{2^e+e-2} - 3\cdot 2^{2^e-3} + 2^{2^{e-1}-2}$.

The total numbers of (i),(ii),(iii) are

(original)

$$2^{2^e} + (2^{2^e-1} - 1) + (2^e - 1)2^{2^e-1} = 2^{2^e+e-1} + 2^{2^e} - 1$$

(reduction by Remark 1)

$$\frac{1}{2}2^{2^e} + (2^{2^e-1} - 1) + \frac{1}{2}(2^e - 1)2^{2^e-1} = 2^{2^e+e-2} + \frac{3}{4}2^{2^e} - 1$$

(reduction by Appendix all)

$$\tfrac{1}{2}2^{2^e} + [2^{(2^e-3)} + 3 \cdot 2^{(2^{e-1}-2)} - 1] + [2^{2^e+e-2} - 3 \cdot 2^{2^e-3} + 2^{2^{e-1}-2}]$$

$$= 2^{2^e+e-2} + 2^{2^e-2} + 2^{2^{e-1}} - 1$$